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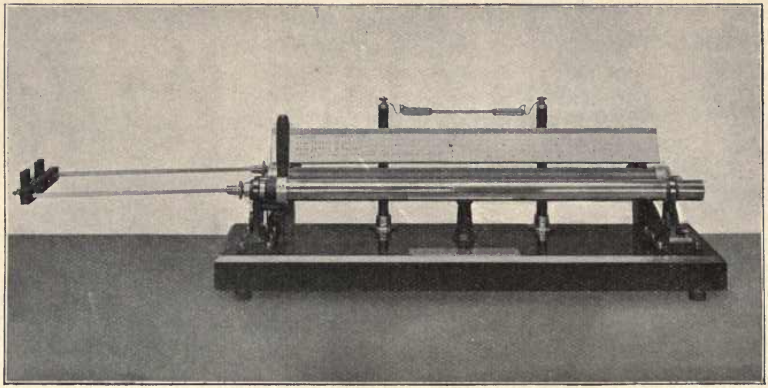

THE FLEMING CYMOMETER.

Frontispiece. See p. 171.

The

Wireless Telegraphist's Pocket Book

of

Notes, Formulæ, and Calculations

BY

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'Cantor Lectures on Hertzian Wave Telegraphy,' etc. etc.

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FOREWORD

THIS little book makes no pretence to be a complete treatise on Wireless Telegraphy, for these latter already exist in considerable numbers. It is merely a *Memoria Technica* of formulæ, data, and useful tables required in practical calculations connected with radiotelegraphy. For the sake of operators and students out of reach of mathematical books, or who have allowed their knowledge to become rusty, the first chapter is occupied with simple mathematical information which will provide the reader with the necessary working tools for subsequent use. Numerical examples are freely introduced because a formula which cannot be applied in arithmetic calculations is useless to the practical electrician.

For this reason Logarithmic Tables, Trigonometrical Tables, Tables of Squares, Cubes, Square Roots, and Cube Roots are given in the last chapter, and these have been as far as possible verified for accuracy.

It has not been found possible, consistently with the limitation of the size of the book to dimensions justifying its name as a pocket book, to give detailed descriptions of either measuring instruments or radiotelegraphic apparatus. It is assumed that the student or reader will have at hand some treatise on Wireless Telegraphy furnishing this information. The function of the present pocket book is merely to provide, in a form handy for reference and assisted by numerical

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examples, the chief formulæ required in electrical calculations connected with alternating high frequency currents and radiotelegraphic work.

For information on the purely engineering side of radiotelegraphy connected with boilers, engines, steam piping, foundations, and building work, the existing engineering pocket books provide; but there seemed to be an opening for a small reference book of radiotelegraphic formulæ which the author has endeavoured to fill.

J. A. F.

University College, London,
May 1915.

LIST OF ERRATA

The reader is requested to make the following corrections—

p. 5 —In the third row of the determinant just below line 1 *instead of* $a_3\ b_3\ c_2$ *read* $a_3\ b_3\ c_3$.

p. 12 —In equation (30) right hand side *instead of*—
$\sqrt{(a^2+b^2)(c^2+a^2)}$ *read* $\sqrt{(a^2+b^2)(c^2+d^2)}$.

p. 13—In line 5 from top in the denominator of the expression for the vector size *instead of* $\sqrt{(c^2+a^2)}$ *read* $\sqrt{(c^2+a^2)}$, and in equation (37) in the vector size make the same correction.

p. 14—In line 9 from top *instead of* $\tan^{-1}\frac{3}{4}$ *read* $\tan^{-1}\frac{4}{3}$

p. 41—In equation (136) *instead of* $y=f(x-ct)+f(x-ct)$ *read* $y=f(x-ct)+f(x+ct)$, and in seventh line below equation (136) *instead of* t by t *read* t by t^1, and in the line below the last *instead of* (128) *read* (136).

p. 45—In equation (149) *instead of*—
$$\mathbf{F}=\mathbf{i}dF_1+\mathbf{j}dF_2+\mathbf{k}dF_3$$
read $\mathbf{F}=\mathbf{i}F_1+\mathbf{j}F_2+\mathbf{k}F_3$.

p. 52—In formula (171) insert after *Cos x* the equality sign =.

p. 72—Line 22 from top *instead of* suspended per second—
read expended per second.

p. 140—In equation (38) right hand side *read* $S=C^2p^2R$.

p. 158—In equation (16) *instead of* $\left(\frac{M}{C}R-P\right)$ *read* $\left(\frac{M}{C}-RP.\right)$

p. 165.—Line 8 from bottom *instead of* $\epsilon^{jpt}=$ *read* $\epsilon^{jPt}=$

Line 9 from bottom *instead of* ϵ^{jpt} *read* ϵ^{jPt}

p. 202—In Section 7 line 5 *instead of* frequency L,—
read frequency N.

p. 210—Line 4 from bottom of page *instead of* tota—
read total.

p. 249—In the diagrams Figs. 35 and 36 the telephone T should be shown placed in the wire connecting the filament of the valve V with the condenser C, and not in the wire connecting the plate G with the condenser C.

p. 268—Line 25 from top *instead of* 2 R.W.—
read 2 K.W.

p. 346—In the Index, first sentence in letter T, supply after phrase Table of Dielectic Constants, the number. 339 for page reference.

CONTENTS

CHAPTER I

PAGE

MATHEMATICAL NOTES 1

1. Mathematical Functions. 2. Linear Equations. 3. Determinants. 4. Scalar and Vector Quantities. 5. Complex Quantities. 6. Scalar and Vector Products of Two Vectors. 7. Algebraic Series. 8. Logarithms. 9. Angular Measurement. 10. Trigonometrical Functions. 11. Differential Calculus. 12. Integral Calculus. 13. Determination of Lengths of Curves, Areas, and Volumes. 14. Second and Higher Differentials, Expansion in Series. 15. Differential Equations. 16. Vector Analysis. 17. Scalar and Vector Potentials. 18. Line, Surface, and Volume Integrals. 19. Hyperbolic Functions.

CHAPTER II

UNITS, DIMENSIONS, AND SYSTEMS OF MEASUREMENT 55

1. Fundamental Units. 2. Derived Units, Velocity, Acceleration, Force, Energy, and Power. 3. Electric and Magnetic Units, Dimensional Expressions for various Units on the Two Systems. 4. Practical Units. 5. The Definition of the Practical Units for Electrical Measurement.

CHAPTER III

HIGH FREQUENCY RESISTANCE AND INDUCTANCE MEASUREMENT 77

1. Difference between High Frequency Resistance and Steady Resistance. 2. Penetration of a Current into a Conductor. 3. Calculation of High Frequency Resistance of Wires. 4. Measurement of High Frequency Resistance of Wires. 5. Inductance of Wires and Circuits, Mutual Inductance, Geometric Mean Distance, Mutual and Self Inductance of Rectangular and Square Circuits and of Straight Parallel Wire, Circular and Spiral Wire Circuits

CHAPTER IV

PAGE

HIGH FREQUENCY CURRENT AND VOLTAGE MEASUREMENT 101

1. The Root-Mean-Square Value of a Simple Periodic Current and of a Damped Oscillation. The Logarithmic Decrement of an Oscillation. 2. Hot Wire Ammeters for the Measurement of High Frequency Currents. 3. Measurement of High Frequency Voltage. Spark Voltages. 4. Relation of Current and Voltage in the case of a Condenser charged with an Alternating Current. 5. Current Flow through Circuits having Inductance, Resistance, and Capacity. 6. Time Period of Electric Oscillation in Circuits having Inductance, Resistance, and Capacity; the Oscillation Constant of a Circuit.

CHAPTER V

CAPACITY MEASUREMENT AND PREDETERMINATION. 124

1. Definition of Electric Capacity. 2. Predetermination of Capacity in certain cases, Sphere, Concentric Spheres, Cylinders, Concentric Cylinders, Multiple Wire Antennæ, Flat Plates. 3. Experimental Determination of Capacities. 4. Addition of Capacities. 5. Dissipation of Energy in Dielectrics of Condensers. 6. Measurements of Energy Losses in Condensers with Alternating Currents, Capacity Measurements with the Cymometer.

CHAPTER VI

BRIDGE METHODS OF MEASUREMENT OF RESISTANCE AND INDUCTANCE 146

1. Networks of Conductors, Bridges, and Potentiometers; the Wheatstone's Bridge problem. 2. Bridge methods of Measurement of small Inductances. 3. Measurement of Mutual Inductance. 4. Coefficients of Coupling. 5. General Conditions for Equilibrium in Bridge Measurements when used with Simple Periodic Currents.

CHAPTER VII

HIGH FREQUENCY CYMOMETER MEASUREMENTS, WAVE LENGTHS AND DECREMENTS . . . 163

1. Oscillatory Circuits and Resonance. 2. Frequency Measurement, Coupled Circuits, Resonance Curves, the Cymometer, Measurement of Frequency, Wave Length, Inductance, and Capacity with High Frequency Oscillations. 3. Delineation of a Resonance Curve, Measurement of Decrement, Marconi, and other forms of Decremeter. 4. Energy Losses in Condensers made with Various Dielectrics.

CHAPTER VIII

PAGE
AERIALS AND ELECTRIC RADIATION . . . 185

1. Maxwell's Theory of the Electromagnetic Field; Maxwell's Equations. 2. The Theory of the Hertzian Oscillator. 3. Radiation from a Hertzian Oscillator, Poynting's Theorem. 4. Radiation from Wireless Telegraph Aerials, Radiation Resistance. 5. Radiation Decrement, Formulæ for Radiation Decrement. 6. Radiation from closed or semi-closed Oscillators. 7. Connection between the Radiation Resistance and Decrement, Radiation Efficiency of an Aerial. 8. Practical Determination of Antenna Resistance. 9. Correction necessary in calculation of the Antenna Frequency due to Inductance of Coils in Series with the Antenna.

CHAPTER IX

TRANSMITTERS AND TRANSMITTING CIRCUITS . . 214

1. Spark Transmitters, Essential Elements, Determination of Fundamental Constants, Frequency, Wave Length, Decrement, and Oscillation Constant. Energy Balance Sheet, Efficiency of Transmitter. 2. Arc Transmitters, Characteristic Curve of Arc. 3. High Frequency Alternators, Goldschmidt's Frequency-raising Alternator, Theory of its Action, Efficiency of the Machine.

CHAPTER X

RECEIVING CIRCUITS AND DETECTORS . . . 233

1. Receiving Circuits; Formula for the received Current; Absorption of Energy by the receiving Antenna, Ratio of the Mean Spherical and Horizontal Radiation of a sending Antenna. 2. Tuners and Jiggers, Marconi Tuner. 3. Detectors for Damped and Undamped Oscillations, Magnetic, Rectifying, and Ionised Gas Detectors; Fleming Oscillation Valve, Poulsen Tikker, Fessenden Heterodyne, Goldschmidt Tone Wheel; Rectifying Detectors, Characteristic Curves, Unilateral Conductivity of certain Pairs of Substances in Contact.

CHAPTER XI

PRACTICAL INFORMATION FOR WIRELESS OPERATORS 255

1. Codes and Signals. 2. Instructions for the Charging and Maintenance of Secondary Batteries. 3. The Care of Wireless Telegraph Apparatus, Induction Coils; Transformers, Spark Dischargers, Condensers, Detectors. 4. Glossary of Electrical Terms.

CHAPTER XII

PAGE

MATHEMATICAL AND PHYSICAL TABLES . . . 282

1. Tables of Squares, Square Roots, Cubes, and Cube Roots of Numbers from 1 to 1000. 2. Table of Ordinary Logarithms from 1 to 1000. 3. Napierian Logarithms from 1 to 100. 4. Table of Natural Sines and Cosines. 5. Table of Hyperbolic Functions. 6. Table of Wire Gauges. 7. Table of Sizes, Weights and Resistances of Copper Wires. 8. Table of Sizes, Weights, and Resistances of Aluminium Wires. 9. Table of Data of Copper and Aluminium. 10. Volume Resistivities of Pure Metals. 11. Volume Resistivities of some Alloys. 12, 13, 14. Tables of Dielectric Constants of Solids, Liquids, and Gases.

INDEX 343

CHAPTER I

MATHEMATICAL NOTES

1. Mathematical Functions

LET a curve of any kind be drawn on a plane (see Fig. 1), and a pair of rectangular axes be drawn in the same plane. Let distances reckoned along the horizontal axis from the intersection of these axes be called *abscissæ* and distances reckoned along, or parallel to, the vertical axis, *ordinates*.

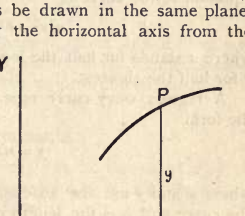


FIG. 1.

Then the abscissa and ordinate of any point P on the curve, reckoned from the intersection O of the axes, taken as the *origin*, are called the *co-ordinates* of that point on the curve, and are denoted by the letters x and y. At every point of the curve there is a certain relation between x and y which is expressed by saying that the ordinate y is a function of the abscissa x. This is written

$$y = f(x), \quad . \quad . \quad . \quad (1)$$

where the f stands for "function of."

If the curve has any regularity it will be possible to express y in the form of some algebraic or trigonometrical or other mathematical expression involving x and certain constants. Thus, to take the simplest case, let the curve be a straight line; then the relation between the co-ordinates of any point on the line can be expressed in the form

$$y = mx + l, \quad . \quad . \quad . \quad (2)$$

where l is the length of the intercept on the y-axis or distance between the origin and the point where the line cuts the y-axis, and m is the tangent of the angle of inclination of the line with respect to the x-axis.

This expression (2) is called the *equation* of the line. It may also be written in the form

$$ax + by = c, \quad . \quad . \quad . \quad (3)$$

where $\frac{c}{b}$ denotes the intercept of the line on the axis OY and $\frac{c}{a}$ the intercept on the axis OX, and hence $\frac{a}{b}$ the tangent of the angle of downward slope of the line.

If the line forms a closed curve such as an ellipse, and the origin of the co-ordinates is taken at the centre, then the equation to the curve is

$$a^2y^2 + b^2x^2 = a^2b^2, \quad . \quad . \quad . \quad (4)$$

where a stands for half the longest diameter of the ellipse and b for half the shortest.

A regular wavy curve repeating itself has an equation of the form

$$y = sin\ \frac{2\pi}{\lambda}x, \quad . \quad . \quad . \quad (5)$$

where x and y are the abscissa and ordinate of any point of the curve and λ is the length of the abscissa comprising one complete cycle of the curve. In such a case there are many values of x, separated by intervals equal to λ, which give the same value to y. This function is therefore called a *periodic function* of x.

2. Linear Equations

An expression of the form $ax + by = c$ is called a *linear equation* because it denotes algebraically a straight line. By an extension of the term all expressions containing only first powers of the variables and certain constants are called linear equations. Thus

$$ax + by + cz + dw = 0 \quad . \quad . \quad (6)$$

is a linear equation, but

$$ax^3 + bx^2 + cx + d = 0 \quad . \quad . \quad (7)$$

is a non-linear equation.

We have frequently to deal with simultaneous linear equations, such for instance as the two equations

$$\begin{aligned} a_1x + b_1y &= c_1, \\ a_2x + b_2y &= c_2, \end{aligned} \quad . \quad . \quad . \quad (8)$$

in which the constants in each equation are different. Under certain conditions we can then find the values of x and y which will satisfy these conditions, provided the constants are given. Each of the equations (8) denotes a straight line. Hence, when we find a value of x and y which satisfies them both simultaneously this is equivalent to finding the co-ordinates of their point of intersection.

In the same way the equations (9)

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3v + c_3z &= d_3 \end{aligned} \quad . \quad . \quad . \quad (9)$$

each denote a straight line drawn in space, and if there are any values of x, y, and z which satisfy them simultaneously, these give the co-ordinates of their point of intersection.

3. Determinants

If we solve the two equations (8) in the usual way by multiplying the first by b_2 and the second by b_1 and subtracting, we obtain the value of x in the form

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}; \quad . \quad . \quad . \quad (10)$$

also we obtain in the same way

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}. \quad . \quad . \quad . \quad (11)$$

Each of the expressions in the numerator and denominator of the values of x and y is called a *determinant*, and is written in the form

$$\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} = a_1b_2 - a_2b_1. \quad . \quad . \quad (12)$$

It will be seen that this determinant consists of four *elements* arranged in two *rows* and two *columns*, and it is written out by cross multiplying the quantities, and thus forming products, each containing one *element* from each row and each column in turn, and prefixing a *plus* sign when we are moving forward,

that is, from first row to second row, and *negative* when we are moving in the reverse direction. The quantity $a_1b_2 - a_2b_1$ is called a *determinant* because it determines whether there can be a simultaneous solution of the equations. Suppose $a_1b_2 - a_2b_1 = 0$ then x and y are both infinite. This is because then $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, and this implies that the straight lines represented by (8) are parallel, and therefore do not intersect, or intersect at infinity.

The values of x and y are therefore each obtained as the quotient of two determinants and we can write them in the form

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \quad . \quad (13)$$

In the same manner we can have a 3-column 3-row determinant formed of 9 elements, written thus

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}. \quad . \quad . \quad . \quad (14)$$

The above stands for

$$aei + bfg + cdh - ceg - bdi - afh. \quad . \quad (15)$$

Each term comprises three factors, viz. an element taken from each column and each row successively. Thus from col. 1 row 1, col. 2 row 2, col. 3 row 3 ; then col. 2 row 1, col. 3 row 2, col. 1 row 3, etc. The negative sign is prefixed when the rows are taken backwards, thus col. 1 row 3, col. 2 row 2, col. 3 row 1, etc. Instead of being formed of algebraic letters a determinant may consist of numbers, and then it can be calculated numerically.

Thus $$\begin{vmatrix} 9 & 1 \\ 8 & 3 \end{vmatrix} = 9 \times 3 - 8 \times 1 = 19, \quad . \quad . \quad (16)$$

and

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 45 + 84 + 96 - 105 - 72 - 48$$
$$= 225 - 225 = 0, \quad . \quad . \quad . \quad (17)$$

where $45 = 1 \times 5 \times 9$, $84 = 2 \times 6 \times 7$, $96 = 3 \times 4 \times 8$.
$105 = 3 \times 5 \times 7$, $72 = 2 \times 4 \times 9$, $48 = 1 \times 6 \times 8$.

If we consider the 3-column determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_2 \end{vmatrix},$$

it is easy to see from the above rule that it can be written

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}. \quad (18)$$

The 2-column determinants entering into the above expression are called the *minors* of the 3-column determinant.

Suppose then that we consider the three simultaneous equations (9). Multiply the first by $(b_2c_3 - b_3c_2)$, the second by $(b_3c_1 - b_1c_3)$, and the third by $(b_1c_2 - b_2c_1)$, and add the results. It will be found that we obtain a result equivalent to

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}. \quad . \quad . \quad . \quad . \quad (19)$$

Also we can find

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad . \quad . \quad . \quad . \quad (20)$$

and

$$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}. \quad . \quad . \quad . \quad . \quad (21)$$

Note that the d_1, d_2, d_3 come in the first, second, or third column in the top determinant, according as we are solving for x, y, or z.

It is often convenient to abreviate a 3- or more column

determinant by writing down merely the dexter diagonal or right-hand diagonal; thus the determinant in the denominator of all the three previous expressions (19), (20), (21) may be written $(a_1b_2c_3)$ and the solution of equations (9) is $x=(d_1b_2c_3)/(a_1b_2c_3)$, $y=(a_1d_2c_3)/(a_1b_2c_3)$, $z=(a_1b_2d_3)/(a_1b_2c_3)$.

If the nine constants a_1, b_1, c_1, etc., are given as numbers, we can at once then write down the solutions of these equations and find the values of x, y, z. Thus if we have three simultaneous equations given, such as

$$\begin{aligned} 7x+6y+2z&=25\\ 3x+4y+5z&=26 \quad . \quad . \quad . \quad (22)\\ 2x+\ y+4z&=16 \end{aligned}$$

We have for the values of x, y, and z the quotients of pairs of determinants which can be written down at once by comparing the results given in (19), (20), (21), as the solution of equations (9).

$$x=\frac{\begin{vmatrix} 25 & 6 & 2\\ 26 & 4 & 5\\ 16 & 1 & 4 \end{vmatrix}}{\begin{vmatrix} 7 & 6 & 2\\ 3 & 4 & 5\\ 2 & 1 & 4 \end{vmatrix}}=\frac{55}{55}=1,$$

$$y=\frac{\begin{vmatrix} 7 & 25 & 2\\ 3 & 26 & 5\\ 2 & 16 & 4 \end{vmatrix}}{\begin{vmatrix} 7 & 6 & 2\\ 3 & 4 & 5\\ 2 & 1 & 4 \end{vmatrix}}=\frac{110}{55}=2,$$

$$z=\frac{\begin{vmatrix} 7 & 6 & 25\\ 3 & 4 & 26\\ 2 & 1 & 16 \end{vmatrix}}{\begin{vmatrix} 7 & 6 & 2\\ 3 & 4 & 5\\ 2 & 1 & 4 \end{vmatrix}}=\frac{165}{55}=3.$$

The reduction and calculation of numerical determinants is facilitated by the four following rules:—

Rule 1.—The value of a determinant is not altered if we change columns into rows or *vice versa*.

Thus, the value of $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

is the same as $\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$.

This can be proved easily by writing out both the above determinants.

Rule 2.—The interchange of two adjacent rows or two adjacent columns reverses the sign of the determinant.

Thus $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = - \begin{vmatrix} b & a \\ d & c \end{vmatrix}$.

This is evident; since $(ad - bc) = -(bc - ad)$.

Also since

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - fh) + d(hc - bi) + g(bf - ec),$$

and since the sign of the terms in the brackets called the minors, viz. $(ei - fh)$, $(hc - bi)$, $(bf - ec)$ is reversed when the last two columns exchange places, it follows that

$$\begin{vmatrix} a & c & b \\ d & f & e \\ g & i & h \end{vmatrix} = - \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix},$$

and the same for any other pair of rows or columns reversed, and for any other determinant.

Rule 3.—If two adjacent rows or two adjacent columns are identical the determinant is zero.

For if two adjacent rows or columns in any determinant are identical, then reversal or interchange must reverse the sign of the determinant. Since, however, the determinant is not altered by interchanging two identical rows or columns this can only be because the said determinant is zero.

Rule 4.—If each term of any column of a determinant is multiplied by the same factor, positive or negative, and then added to the corresponding terms of any other column, the value of the determinant is unaltered. Thus, if we take

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}, \quad . \quad . \quad . \quad (23)$$

and subtract the second column from the first, and write these differences as a new first column, we obtain

$$\begin{vmatrix} a-b, & b, & c \\ d-e, & e, & f \\ g-h, & h, & i \end{vmatrix} \quad . \quad . \quad . \quad (24)$$

This is equivalent to multiplying the second column all through by -1 and adding it to the first.

Now (24) is the same as

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} b & b & c \\ e & e & f \\ h & h & i \end{vmatrix} \quad . \quad . \quad . \quad (25)$$

But the last determinant in (25) has two identical columns. Hence by Rule 3 its value is zero. Hence (24) is equal to (23).

This last rule is of great value in enabling us to evaluate quickly a 3- or more column determinant. Thus, take the determinant

$$\begin{vmatrix} 7 & 6 & 2 \\ 3 & 4 & 5 \\ 2 & 1 & 4 \end{vmatrix},$$

which we have seen is equal to 55. Subtract the second column from the first column and put these differences as a new first column, and we have

$$\begin{vmatrix} 1 & 6 & 2 \\ -1 & 4 & 5 \\ 1 & 1 & 4 \end{vmatrix}.$$

Now add the first row to the second row for a new second row, and subtract the first row from the third for a new third row, and we obtain

$$\begin{vmatrix} 1 & 6 & 2 \\ 0 & 10 & 7 \\ 0 & -5 & 2 \end{vmatrix}.$$

This last determinant is equal to $\begin{vmatrix} 10 & 7 \\ -5 & 2 \end{vmatrix}$, which is 55.

In this manner we avoid the use of many multiplications, and by a skilful use of Rule 4 we can reduce a determinant to a form in which it has the final element of its first column

and first row unity, and all the other terms in the first column zero.

4. Scalar and Vector Quantities

We are concerned in mathematics with two kinds of quantities: one, called *scalar*, which are adequately defined by a single number or statement of magnitude, such as *a mass*, *a temperature*, *a pressure*, or *a potential*. The other class of quantity are called *vectors*. These are directed quantities, and require two numbers to define them, one defining their *size* or *magnitude*, and the other their *direction* or *phase*. Such quantities are *forces*, *velocities*, *magnetic* or *electric forces*. Also *alternating electric currents* or *electromotive forces* are vector quantities.

The scalar quantities can be denoted by a single algebraic symbol, such as a or x; but the vector quantities need to be denoted by some symbol which connotes direction as well as magnitude.

By a very general convention a vector is denoted by a letter printed or written in thick type, *e.g.* clarendon (**E**) or block ($\mathbb{E}$). The mere size or magnitude is denoted by a similar letter in Roman or italic type (E).

Corresponding to these two quantities we have two kinds of addition.

The addition of scalars merely involves the arithmetic addition of their magnitudes. In the case of vectors addition implies the idea of joint effect. Hence vectors are added by the law of the parallelogram of forces, and not by the mere arithmetic addition of their sizes. Thus the vector equation

$$\mathbf{A}+\mathbf{B}=\mathbf{C} \quad . \quad . \quad . \quad (26)$$

means that a vector denoted by **A**, and therefore capable of being represented by a straight line, is added to another vector denoted by **B** by drawing a pair of lines of length A and B as adjacent sides of a parallelogram, which represent these vectors in direction as well as magnitude, and then their joint effect or vector sum is represented by the diagonal of that parallelogram which passes through the intersection of the two lines **A** and **B**.

Another mode of representing a vector is as the vector sum of its components or projections along three rectangular axes. Let three axes Ox, Oy, Oz be drawn at right angles (see Fig. 2), and from the origin let any vector $OP=\mathbf{r}$ be drawn. Let OX, OY, OZ be the projections of OP on the

three axes. Let $OX = \mathbf{i}x$, $OY = \mathbf{j}y$, $OZ = \mathbf{k}z$ where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors set off along the axes and x, y, and z represent the length of OX, OY, OZ. Then, since OP is the vector sum of the projections, we can write a vector equation

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z. \quad (27)$$

The size of the vector $\mathbf{r}$ may be denoted by r, and hence

$$r^2 = x^2 + y^2 + z^2. \quad (28)$$

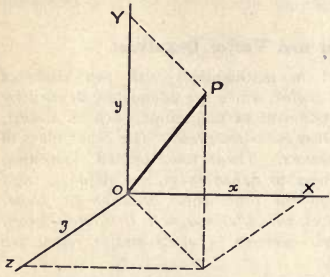


FIG. 2.

5. Complex Quantities

Another mode of representing a vector is by a *complex quantity*. If $\mathbf{a}$ denotes a horizontal vector or line drawn from an origin horizontally towards the right hand then $-\mathbf{a}$ would represent a line drawn horizontally from the origin towards the left hand.

We may regard $-\mathbf{a}$ as obtained from $\mathbf{a}$ by the process of rotating the vector round its left-hand extremity through 180° or two right angles.

Also we may regard $-\mathbf{a}$ as obtained from $\mathbf{a}$ by operating on it by -1.

Hence on this convention -1 applied to a vector rotates it through two right angles. We may then ask what symbol or operator may be regarded as rotating it through one right angle or through 90°.

The answer is, it must be such a symbol as twice repeated will yield -1.

Now $\sqrt{-1}$ is such a symbol; because $\sqrt{-1} \times \sqrt{-1} = -1$.

Accordingly $\sqrt{-1}$, denoted usually by j, may be considered as an *operator* turning a vector through one right angle.

Hence, if $\mathbf{a}$ represents a line of length a drawn horizontally, $j\mathbf{a}$ will represent a line of length a drawn vertically. Also if $\mathbf{a}$ represents a line of length a drawn horizontally and $j\mathbf{b}$ a line of length b drawn vertically at the right-hand extremity of $\mathbf{a}$, then $\mathbf{a} + j\mathbf{b}$ represents the vector sum of $\mathbf{a}$ and $\mathbf{b}$ and is therefore a proper mode of representing the hypothenuse of

the right-angled triangle whose sides are a and b. The length of this hypotenuse is $\sqrt{a^2+b^2}$.

Quantities such as $\mathbf{a}+j\mathbf{b}$ or $\mathbf{a}+\sqrt{-1}\mathbf{b}$ are called complex quantities and $\sqrt{a^2+b^2}$ is called the *size* or *modulus* of $\mathbf{a}+j\mathbf{b}$.

We meet with such complex quantities as the result of solving certain quadratic equations.

There are certain important rules for the addition, multiplication, and division of complex quantities and their sizes or moduli as follows:

The first term or symbol not multiplied by $j=\sqrt{-1}$ is called the *real part* or *horizontal step* of the complex. The second term or part multiplied by j is called the *imaginary part* or *vertical step* of the complex.

Rule 1.—To add together two complexes, add the horizontal steps for a resultant horizontal step and the vertical steps for a resultant vertical step. Thus

$$\mathbf{a}+j\mathbf{b}+\mathbf{c}+j\mathbf{d}=(\mathbf{a}+\mathbf{c})+j\,(\mathbf{b}+\mathbf{d}).$$

The proof of this is easily seen by drawing on a plane a line to represent $\mathbf{a}+j\mathbf{b}$ and then from its right-hand extremity another line to represent $\mathbf{c}+j\mathbf{d}$. If we then draw a pair of rectangular axes through the left-hand extremity of the first vector and project both vectors on to these axes (see Fig. 3), it is easy to see that the horizontal projection of the vector sum of the two vectors, that is the diagonal of the parallelogram of which these vectors form adjacent sides, is $\mathbf{a}+\mathbf{c}$, whilst the vertical step is $\mathbf{b}+\mathbf{d}$. Hence it follows that the sum of the two vectors is $(\mathbf{a}+\mathbf{c})+j(\mathbf{b}+\mathbf{d})$.

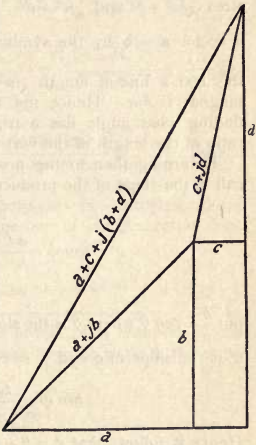


FIG. 3.

It should be noted that $\mathbf{a}+j\mathbf{b}$ stands not only for one particular line or vector but for any line of equal length and identical direction parallel to it. Hence, if two vectors are equal, or if $\mathbf{a}+j\mathbf{b}=\mathbf{c}+j\mathbf{d}$, we can equate respectively the horizontal and the vertical steps, viz. $\mathbf{a}=\mathbf{c}$ and $\mathbf{b}=\mathbf{d}$.

Rule 2.—To multiply two complexes we multiply the sizes and add the slopes. The slope of a vector is the angle it makes with the horizontal line. If θ denotes this angle and $\mathbf{a}+j\mathbf{b}$ the vector, then

$$tan\ \theta = b/a \text{ or } \theta = tan^{-1}\frac{b}{a}.$$

The size of the vector is $\sqrt{a^2+b^2}$. If then there be two complexes $\mathbf{a}+j\mathbf{b}$ and $\mathbf{c}+j\mathbf{d}$ their product is

$$(\mathbf{ac}-\mathbf{bd})+j(\mathbf{ad}+\mathbf{bc}). \quad . \quad . \quad (29)$$

The size of this last complex is

$$\sqrt{(ac-bd)^2+(ad+bc)^2} = \sqrt{(a^2+b^2)(c^2+a^2)}. \quad (30)$$

Hence the size of the product is the product of the respective sizes $\sqrt{a^2+b^2}$ and $\sqrt{c^2+d^2}$. It is clear that we can represent a vector $\mathbf{a}+j\mathbf{b}$ by the symbol $\sqrt{a^2+b^2}\big/tan^{-1}\frac{b}{a}$, meaning by this last a line of length $\sqrt{a^2+b^2}$ tilted up at an angle whose tangent is b/a. Hence the slope of a vector is obtained by finding what angle has a trigonometrical angle equal to the ratio of the length of the vertical to the horizontal step.

Returning then to the product of our two vectors. If we call ϕ the slope of the product, then clearly from (29)

$$tan\ \phi = \frac{ad+bc}{ac-bd} = \frac{\frac{d}{c}+\frac{b}{a}}{1-\frac{b}{a}\frac{d}{c}}; \quad . \quad . \quad (31)$$

but $\frac{b}{a} = tan\ \theta$ where θ is the slope of $\mathbf{a}+j\mathbf{b}$, and $\frac{d}{c} = tan\ \theta'$ where θ' is the slope of $\mathbf{c}+j\mathbf{d}$. Accordingly,

$$tan\ \phi = \frac{tan\ \theta' + tan\ \theta}{1 - tan\ \theta\ tan\ \theta'}. \quad . \quad . \quad (32)$$

Hence it follows that $\phi = \theta + \theta'$, or the slope of the product is the sum of the slopes of the two vectors.

Accordingly we can write the product of $\mathbf{a}+j\mathbf{b}$ and $\mathbf{c}+j\mathbf{d}$ in the form

$$\sqrt{a^2+b^2}\ \sqrt{c^2+a^2}\Big/tan^{-1}\frac{b}{a}+tan^{-1}\frac{d}{c}, \quad . \quad (33)$$

which signifies that the size of the product is the product of the

sizes of the respective vectors, and that the slope of the product is the sum of the slopes of the respective vectors.

Rule 3.—To divide two complexes we take the quotient of the sizes and subtract the slopes.

Thus $\frac{\mathbf{a}+j\mathbf{b}}{\mathbf{c}+j\mathbf{d}}$ has a size equal to $\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+a^2}}$ and a slope ϕ such that

$$\phi = tan^{-1}\frac{b}{a} - tan^{-1}\frac{d}{c}.$$

To prove this, multiply both numerator and denominator of $\frac{\mathbf{a}+j\mathbf{b}}{\mathbf{c}+j\mathbf{d}}$ by $\mathbf{c}-j\mathbf{d}$.

We obtain
$$\frac{ac+bd}{c^2+d^2} + j\frac{bc-ad}{c^2+d^2}. \quad . \quad . \quad (34)$$

The size of the last complex is

$$\sqrt{\frac{(ac+bd)^2}{c^2+d^2} + \frac{(bc-ad)^2}{c^2+a^2}} = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}. \quad . \quad (35)$$

Also the slope ϕ is such that

$$tan\,\phi = \frac{bc-ad}{ac+bd} = \frac{tan\,\theta - tan\,\theta'}{1+tan\,\theta\,tan\,\theta'}. \quad . \quad (36)$$

Hence $\phi = \theta - \theta'$.

Hence the quotient of two complexes is a complex, whose size is obtained by taking the quotient of the respective sizes and subtracting their slopes, or

$$\frac{\mathbf{a}+j\mathbf{b}}{\mathbf{c}+j\mathbf{d}} = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+a^2}} \Big/ tan^{-1}\frac{b}{a} - tan^{-1}\frac{d}{c}. \quad . \quad (37)$$

Rule 4.—The size of the reciprocal of a complex is the reciprocal of the size of the original, and the slope is the same angle but negative.

Thus the size of $\frac{1}{\mathbf{a}+j\mathbf{b}}$ is $\frac{1}{\sqrt{a^2+b^2}}$.

To prove this multiply $\frac{1}{\mathbf{a}+j\mathbf{b}}$ top and bottom by $\mathbf{a}-j\mathbf{b}$, and we obtain $\frac{\mathbf{a}}{\mathbf{a}^2+\mathbf{b}^2} - j\frac{\mathbf{b}}{\mathbf{a}^2+\mathbf{b}^2}$. But the size of this last is

$$\sqrt{\left(\frac{a}{a^2+b^2}\right)^2 + \left(\frac{b}{a^2+b^2}\right)^2} = \frac{1}{\sqrt{a^2+b^2}}. \quad (38)$$

The slope of the vector $\frac{1}{\mathbf{a}+j\mathbf{b}}$ is therefore an angle ϕ such that $tan\ \phi = -\frac{b}{a}$ and this is equal to the slope of $\mathbf{a}+j\mathbf{b}$ but is negative or in the reverse direction.

These rules will be best appreciated by applying them to a numerical example. Thus

Example.—Plot the vectors $3+j4$ and $1+j1$.

Answer.—The first is a line with a horizontal projection of 3 units and a vertical one of 4 units. Its length or size is therefore $\sqrt{3^2+4^2}=5$ and its slope $\phi = tan^{-1}\frac{3}{4} = tan^{-1}\ 1.333 = 53°.7'30''$.

The second is a line with horizontal and vertical projections of 1 unit.

Its size is therefore $\sqrt{2}=1.414$ units and its slope $\phi = tan^{-1}1 = 45°$.

Add, subtract, multiply, and divide the complexes or vectors $3+j4$, $1+j1$.

Answer.—$4+j5$ is the sum.

$2+j3$ is the difference.

$7.07 \angle 98°\ 7'\ 30''$ is the product.

$3.53 \angle 8°\ 7'\ 30''$ is the quotient.

Since $\mathbf{a}+j\mathbf{b}$ is equal to

$$\sqrt{a^2+b^2}\ cos\ \phi + j\sqrt{a^2+b^2}\ sin\ \phi,$$

where $\tan\ \phi = \frac{b}{a}$, we can write the last two numerical complexes in the form

$$-7.07\ cos\ (81°\ 52'\ 30'') + j7.07\ sin\ (81°\ 52'\ 30'');$$

$$3.52\ cos\ (\ 8°\ \ 7'\ 30'') + j3.52\ sin\ (\ 8°\ \ 7'\ 30''),$$

which are respectively $-1+j7$ and $3.48+j0.49$.

Hence

$$\begin{aligned}
&(3+j4)+(1+j1) = 4+j5\\
&(3+j4)-(1+j1) = 2+j3\\
&(3+j4)\times(1+j1) = -1+j7\\
&(3+j4)\div(1+j1) = 3.48+j0.49.
\end{aligned} \qquad \ldots \quad (39)$$

6. Scalar and Vector Products of Two Vectors

In addition to the mode of multiplying two vectors represented by complex quantities explained in the previous section, there are two other kinds of products of vectors called respectively the *scalar* and *vector* products.

The scalar product of two vectors is defined to be the product of their sizes multiplied by the cosine of the angle θ between them. Thus if $\mathbf{A}$ and $\mathbf{B}$ are two vectors, and A and B are their sizes, the scalar product is $AB \cos \theta$. There are various ways of denoting this product, but the simplest is the use of a *dot* between the vector symbols. Thus we have

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta. \quad . \quad . \quad . \quad (40)$$

This product is a scalar quantity. It is the numerical product of the length of one vector and the projected length of the other vector on it. The vector product of two vectors is the product of their sizes and the sine of the angle between them. This is best denoted by a small cross put between the vector symbols. Thus

$$\mathbf{A} \times \mathbf{B} = AB \sin \theta. \quad . \quad . \quad (41)$$

Oliver Heaviside denotes this vector product by a letter V placed before the vector symbols, thus: $V\mathbf{AB}$; but this is not so convenient a notation as the small cross, as the letter V is apt to be mistaken for a quantity. The vector product of two vectors is a vector, and it is denoted by a line of length equal to $AB \sin \theta$, drawn perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$ and in such a direction that the twist necessary to turn the first-named vector to coincide with the second is related to the direction of the vector product vector as the twist and thrust of a British corkscrew.

Hence it follows that whereas in the scalar product the order of the vector symbols is indifferent, in the case of the vector product a reversal of the order of the symbols reverses the sign of the product. Thus $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$, but

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}. \quad . \quad . \quad (42)$$

If we consider three vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, of unit length and all at right angles to each other, that is directed along three axes Ox, Oy, Oz, we see that the scalar products taken pair and pair are as follows:

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1, \text{ since } \cos O = 1,$$

and the sizes of **i**, **j**, and **k** are all unity. But any other pairs of scalar products,

$$\mathbf{i}\cdot\mathbf{j},\ \mathbf{j}\cdot\mathbf{k},\ \mathbf{k}\cdot\mathbf{i},\ \mathbf{j}\cdot\mathbf{i},\ \mathbf{k}\cdot\mathbf{j},\ \mathbf{i}\cdot\mathbf{k},$$

are zero.

Apply this to the case of two vectors **r** and **r'** whose axial components are r_1, r_2, r_3, r_1', r_2', r_3'. The scalar product is

$$\mathbf{r}\cdot\mathbf{r}' = (\mathbf{i}r_1 + \mathbf{j}r_2 + \mathbf{k}r_3)\,.\,(\mathbf{i}r_1' + \mathbf{j}r_2' + \mathbf{k}r_3')$$

or

$$\mathbf{r}\cdot\mathbf{r}' = r_1r_1' + r_2r_2' + r_3r_3' \quad . \quad . \quad (43)$$

If, however, we consider vector products then we have $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0$, since $\sin\theta = 0$ in these cases.

Also we have

$$\begin{aligned}\mathbf{j}\times\mathbf{k} &= \mathbf{i} = -\mathbf{k}\times\mathbf{j}\\ \mathbf{k}\times\mathbf{i} &= \mathbf{j} = -\mathbf{i}\times\mathbf{k}.\\ \mathbf{i}\times\mathbf{j} &= \mathbf{k} = -\mathbf{j}\times\mathbf{i}\end{aligned} \quad . \quad . \quad (44)$$

The above rules for the scalar and vector products of **i**, **j**, and **k** are very important. Consider then the vector product of **r** and **r'**. We have

$$\mathbf{r}\times\mathbf{r}' = (\mathbf{i}r_1 + \mathbf{j}r_2 + \mathbf{k}r_3)\times(\mathbf{i}r_1' + \mathbf{j}r_2' + \mathbf{k}r_3'). \quad (45)$$

If the right-hand side is multiplied out, having regard to the proper vector product of the unit vectors, we obtain

$$\mathbf{r}\times\mathbf{r}' = \mathbf{i}(r_2r_3' - r_3r_2') + \mathbf{j}(r_3r_1' - r_1r_3') + \mathbf{k}(r_1r_2' - r_2r_1').$$

The above expression can be written more compactly as a determinant,

$$\mathbf{r}\times\mathbf{r}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_1 & r_2 & r_3 \\ r_1' & r_2' & r_3' \end{vmatrix} \quad . \quad . \quad . \quad (46)$$

Therefore the vector product of two vectors is expressible as a 3-column determinant, and the minors of the elements in the first row are the three axial components of this vector product.

7. Algebraic Series

Any consecutive figures or algebraic quantities which follow a definite law of decrease or increase is called a *Series*. The simplest instance is an *Arithmetic Progression* or series in which there is a constant difference d between adjacent terms.

Thus

$$a,\ a+d,\ a+2d,\ a+3d\ .\ .\ .\ \overline{a+n-1d}\,. \quad (47)$$

is an *Arithmetic Progression.*

It will be noticed that the sum of the first term a and the nth term $(\overline{a+n-1d})$ is $2a+\overline{n-1d}$. Also the sum of the

second term and the $(n-1)$th term is the same. Hence if we take n terms of the series and divide them into two groups of $\frac{n}{2}$ terms we can so arrange them that there are $\frac{n}{2}$ pairs of terms which added together are equal to $2a+\overline{n-1}d$. Hence the sum S of n terms of such a series must be equal to

$$S=\frac{n}{2}\left\{2a+\overline{n-1}d\right\}. \quad . \quad . \quad (48)$$

Thus the sum of the natural numbers 1, 2, 3, etc., up to 100 is

$$S=\frac{100}{2}\left\{2+99\right\}=5050.$$

If in any series of quantities or numbers each one is equal to the preceding one multiplied by a constant factor or number, such a series is called a *Geometric Progression.*

Thus a, ar, ar^2, ar^3, etc. . . . ar^{n-1} . (49)

is a *Geometric Progression.*

Let S_n be the sum of n terms of such a series. Then we have

$$S_n=a+ar+ar^2+ar^3+\text{etc.} \ . \ . \ . \ ar^{n-1}. \ . \ (50)$$

Multiply all through by r, and we have

$$rS_n=ar+ar^2+ar^3+ar^4+ \ . \ . \ . \ ar^n. \quad . \quad (51)$$

Now subtract (50) from (51) and we have

$$S_n(r-1)=a(r^n-1)$$

or $$S_n=a\frac{r^n-1}{r-1} \text{ or } a\frac{1-r^n}{1-r}. \quad . \quad . \quad (52)$$

If r is a number greater than unity then we can only apply the formula (52) to calculate the sum of a finite number of terms.

If, however, r is a fraction, then r^n, when n is very large or infinite, approximates to zero. Hence for a *common ratio* r less than unity we can find the sum of an infinite number of terms of a geometric progression by the formula

$$S=\frac{a}{1-r}. \quad . \quad . \quad . \quad (53)$$

Thus: Find the sum of an infinite number of terms of the series

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\text{etc.} \quad . \quad . \quad (54)$$

Here we have $a=1$, $r=\frac{1}{2}$, and hence $S=2$. Therefore

the sum of an infinite number of terms of (54) continually approaches 2. This is an instance of a *Convergent Series.*

Another geometric progression is the series

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - \text{etc.} \quad . \quad (55)$$

The common ratio in (55) is $r = -1$, and hence

$$S = \frac{1}{1+1} = \frac{1}{2}. \quad . \quad . \quad . \quad (56)$$

Accordingly, the sum of an infinite number of terms of (55) is $\frac{1}{2}$, although the sum of a finite number is 0 or 1, depending upon whether we take an even or an odd number of terms.

Another important expansion is that called the *Binomial Theorem.*

If we square $a + x$ we have

$$(a+x)^2 = a^2 + 2ax + a^2, \quad . \quad . \quad (57)$$

also
$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3. \quad . \quad (58)$$

and
$$(a+x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4. \quad . \quad (59)$$

The general expansion for the nth power is

$$(a+x)^n = a^n + na^{n-1}x + \frac{n\overline{n-1}}{1.2}a^{n-2}x^2 + \frac{n\overline{n-1}\,\overline{n-2}}{1.2.3}a^{n-3}x^3 + \text{etc.} \quad (60)$$

or

$$(1+x)^n = 1 + nx + \frac{n\overline{n-1}}{1.2}x^2 + \frac{n\overline{n-1}\,\overline{n-2}}{1.2.3}x^3 + \text{etc.} \quad (61)$$

The series of number n, $\frac{n.\overline{n-1}}{1.2}$, $\frac{n\overline{n-1}\,\overline{n-2}}{1.2.3}$, etc., are called the *Binomial Coefficients.*

For $n = 2$. They are 1, 2, 1.
For $n = 3$. „ „ 1, 3, 3, 1.
For $n = 4$. „ „ 1, 4, 6, 4, 1.
For $n = 5$. „ „ 1, 5, 10, 10, 5, 1.

If, however, n is negative or a fraction, then the coefficients can never become zero, and hence the series or expansion continues indefinitely. Nevertheless the coefficients are calculated by the same formulæ.

A useful expansion is that of $(a+x)^{\frac{1}{2}}$, which is the same as

$$\sqrt{(a+x)}.$$

The binomial coefficients in this case are

$$1, \tfrac{1}{2}, -\tfrac{1}{8}, \tfrac{1}{16}, -\tfrac{1}{128}, \text{etc.,}$$

and the expansion is

$$\sqrt{(a+x)} = a^{\frac{1}{2}} + \tfrac{1}{2}a^{-\frac{1}{2}}x - \tfrac{1}{8}a^{-\frac{3}{2}}x^2 + \tfrac{1}{16}a^{-\frac{5}{2}}x^3 - \text{etc.} \quad (62)$$

$$= \sqrt{a}\left(1 + \frac{x}{2a} - \frac{x^2}{8a^2} + \frac{x^3}{16a^3} - \frac{5x^4}{128a^4} + \text{etc.}\right). \quad (63)$$

If $\frac{x}{a}$ is a small fraction, then all terms after the second may be neglected, and we have

$$\sqrt{(a+x)} = \sqrt{a} + \frac{x}{2\sqrt{a}}. \quad . \quad . \quad (64)$$

Hence the square root of $1+x$ when x is small compared with unity is $1+\frac{x}{2}$.

Example.—What is the square root of 100.16?

Answer.—10.008.

For $100.16 = 100 + \frac{16}{100}$, and the square root is therefore by (64) $10 + \frac{16}{2000}$, $= 10.008$.

An important expansion is that called the *Exponential Theorem.* It is the expansion in a series of powers of x of the exponential a^x. If we put $1+b$ for a we can make it a case of the binomial theorem and prove that

$$a^x = 1 + Ax + \frac{A^2}{2}x^2 + \frac{A^3}{1.2.3}x^3 + \frac{A^4}{1.2.3.4}x^4 + \text{etc.}, \quad (65)$$

where

$$A = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \text{etc.} \quad (66)$$

If A is made equal to unity then it can be shown that $a = 2.7182818$, and this number is denoted by ϵ.

Hence

$$\epsilon^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \text{etc.} \quad . \quad (67)$$

The products $1.2 = 2$ and $1.2.3 = 6$ and $1.2.3.4 = 24$, etc., are denoted by $2!$, $3!$, $4!$. So the series becomes

$$\epsilon^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \text{etc.} \quad . \quad (68)$$

If $x = 1$ we have

$$\epsilon = 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \text{etc.} = 2.7182818. \quad (69)$$

8. Logarithms

If $a^x = N$ then x is called the logarithm of N to the base a.

There are two systems of logarithms in use, one with the base 10, called the *common* or *Briggs logarithms*, and another with base $\epsilon = 2.7182818$, called the *natural* or *Napierian logarithms*.

Hence if $10^x = N$ and $\epsilon^y = N$ we call x the common logarithm (or *com. log.*) of N and y the Napierian logarithm (*Nap. log.*) of N.

Now if $y = Ax$ where A is some arithmetic factor we have

$$10^x = \epsilon^{Ax}$$

or

$$10 = \epsilon^A.$$

Hence A is the logarithm of 10 to the base ϵ, which is written $A = log_\epsilon 10$.

The reciprocal $M = \frac{1}{A}$ is called the *modulus*.

It can be shown that

$$log\,(N+1) - log\,N = 2M\left\{\frac{1}{2N+1} + \frac{1}{3(2N+1)^3} + \frac{1}{5(2N+1)^5} + \text{etc.}\right\} \quad (70)$$

But both ϵ^0 and $10^0 = 1$.

Hence the logarithm of 1 is zero whichever base is taken.

Accordingly we can calculate *log* 2 by the series (70) by putting N = 1

$$log\,2 = 2M\left(\frac{1}{3} + \frac{1}{3^4} + \frac{1}{5.3^5} + \text{etc. . . .}\right)$$

Napier selected his base so that $M = 1$, and hence

$$log_\epsilon 2 = \frac{2}{3} + \frac{2}{81} + \frac{2}{1215} + \text{etc.} = 0.6931472.$$

Having obtained $log_\epsilon 2$ we can find $log_\epsilon 3$ and so on. Now $log_\epsilon 10 = 2.3025851$.

Hence and

$$\left.\begin{aligned} M &= 0.43429448 \\ A &= 2.3025851 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (71)$$

These two numbers are very important.

Accordingly we obtain the Napierian logarithm by multiplying the common logarithm to base 10 by 2.3025851, or by 2.3026 nearly.

Thus

$$log_{10}\,2 = 0.4343 \times log_\epsilon 2$$
$$= 0.4343 \times 0.69315 = 0.30103.$$

At the end of this Pocket Book will be found a table of common and also of Napierian logarithms (see Chap. XII.).

The use of these logarithms is to enable us to effect multiplication and division by means of addition and subtraction.

For if $10^x = N$ and $10^y = N'$,

then clearly $10^{x+y} = NN'$

and $10^{x-y} = \frac{N}{N'}$

or $x + y = log_{10} NN'$ and $x - y = log_{10} \frac{N}{N'}$.

But $x = log_{10} N$ and $y = log_{10} N'$,

$\therefore log_{10} N + log_{10} N' = log_{10} NN'$

and $log_{10} N - log_{10} N' = log_{10} \frac{N}{N'}$. (72)

Thus we can find by the aid of a table of logarithms the product, or quotient of, two numbers, or any power of that number.

The following are the principal rules to be observed in using a table of ordinary logarithms to the base 10.

Rule 1.—The logarithm of any number consists of two parts, viz. a decimal part called the *mantissa*, which may extend to 4, 5, 6 or 7 places of decimals, and an integer part called the *characteristic*. The characteristic may be a negative integer, but the mantissa is always positive.

Rule 2.—The logarithm of 10 is 1, of 100 is 2, of 1000 is 3, and of any power of 10 it is the corresponding exponent of 10.

For $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, etc.

Hence the characteristic of any number is one less than the number of integer figures which compose it. Thus the characteristic of 3079 is 3, and of 307900 is 5.

In tables of ordinary logarithms only the mantissa is given.

Thus opposite 3079 we find in the tables 0.48841. The user has to supply the characteristic.

Thus the value of log_{10} 3079 is 3.48841, and of 30.79 it is 1.48841, and of 3.079 it is 0.48841.

Also since 0.3079 is $3.079 \div 10$ we have log_{10} $0.3079 = 0.48841 - 1$, which is written $\bar{1}.48841$.

Similarly log_{10} $0.03079 = \bar{2}.48841$.

Rule 3.—To multiply two numbers, look out the *logs* of each and add them. Then find the number which has this sum as its logarithm.

Example.—Multiply 30.79 by 29.76. We find by the

above rules from a table of logarithms $log_{10}\ 30.79 = 1.48841$ and $log_{10}\ 29.76 = 1.47363$. Adding, we have $1.48841 + 1.47363 = 2.96204$. Then look out in the table the number which has 2.96204 for its mantissa, and we find it to be 916.3. Hence this is the product of 30.79 and 29.76 to 1 decimal place. In the same manner, by subtracting the logarithms of two numbers we can find the quotient of one by the other. Also we can find the value of any power of a number as follows :—

Rule 4.—To find a power of any number, look out the logarithm of that number and multiply it by the number representing the required power. Then look out the number which has this product for its logarithm.

Example.—Find the value of $(2.718)^8$. We find $\log_{10} 2.718 = 0.43425$ and hence $3 \times 0.43425 = 1.30275$. Now $\log_{10} 20.08 = 1.30276$. Hence $(2.718)^8 = 20.08$ nearly.

Rule 5.—To find the Napierian logarithm of any number, find the ordinary logarithm and multiply it by 2.3026.

Example.—Find $\log_e 12.30$. We find $\log_{10} 12.30 = 1.08991$. Hence $\log_e 12.30 = 2.3026 \times 1.08991 = 2.50963$.

9. Angular Measurement

In ordinary work the inclination of two straight lines or an angle is measured in degrees, minutes and seconds.

The circumference of a circle is divided into 360 degrees, each degree into 60 minutes, and each minute into 60 seconds denoted by $^\circ\ '\ ''$. Thus $85^\circ\ 17'\ 30''$ signifies 85 degrees 17 minutes and 30 seconds.

We may then consider a pair of radii *OP*, *OQ* drawn from the centre of the circle. They cut off a certain sector *PQ* from the circular circumference (see Fig. 3). This sector reckoned in degrees, minutes and seconds is the measure of the angle which these radii make with each other.

On this system two lines at right angles make an angle of 90° and two lines in one straight line an angle of 180°.

In theoretical work an angle is measured in *radians*. An angle of 1 radian is the angle made by two radii of a circle which cut off from the circumference a sector equal in length to the radius. Since the length of the circumference of a circle is 2π times the radius or 2×3.14159265 times the radius in length, it follows that an angle of 90° in ordinary measure is equal to $\frac{\pi}{2}$ radians or to 1.57079 radians.

Hence 1 radian is an angle of $\frac{180}{\pi}$ degrees or to 57.29578 degrees or 57° 17′ 45″ nearly. This is called generally a *unit angle in circular measure.* Hence the following rules are useful :—

Rule 1.—To convert an angle measured in degrees, minutes and seconds into radians, first express the angle in degrees and decimals of a degree and then divide by 57.29578.

Rule 2.—To convert an angle reckoned in radians into degrees, minutes and seconds, multiply 57° 17′ 45″ by the circular measure of the angle.

Since 1 degree = 60 minutes, 1 minute = 0.016666 of a degree, also since 1 degree = 3600 seconds, 1 second = 0.00027777 of a degree.

10. Trigonometrical Functions

If two straight lines *OP*, *OM* (see Fig. 4) make any angle with each other, one being drawn in a horizontal direction, and

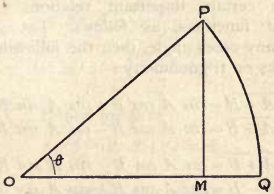


FIG. 4.

if from the extremity *P* of the sloping line a perpendicular *PM* is dropped upon the other, then the ratio

$\frac{PM}{OP}$ is called the natural *sine* of the angle *POM*

$\frac{OM}{OP}$ " " *cosine* " "

$\frac{PM}{OM}$ " " *tangent* " "

$\frac{OP}{PM}$ " " *cosecant* " "

$\frac{OP}{OM}$ is called the natural *secant* of the angle POM

$\frac{OM}{PM}$ " " *cotangent* " "

Thus, if two lines are drawn at an angle of 30°, it is clear that the perpendicular PM would be equal to one-half of OP, and therefore that for 30° the sine is $\frac{1}{2}$ and the cosine $\frac{\sqrt{3}}{2}$ and the tangent is $\frac{1}{\sqrt{3}}$. Hence $sin\ 30° = 0.5$, $cos\ 30° = 0.866$, $tan\ 30° = 0.577$.

If the angle were 45° we should have the ratio of perpendicular PM to hypothenuse OP equal to $\frac{1}{\sqrt{2}}$ and the same for the ratio OM to OP. Hence $sin\ 45° = 0.707$, $cos\ 45° = 0.707$ and $tan\ 45° = 1$.

Tables of the natural sines, cosines and tangents of various angles are given at the end of this Pocket Book (see Chap. XII.).

There are certain important relations between these trigonometrical functions, as follows. Let A denote any angle and B any other angle, then the following formulæ are proved in books on trigonometry:

$$\left.\begin{aligned} sin\ \overline{A+B} &= sin\ A\ cos\ B + cos\ A\ sin\ B \\ sin\ \overline{A-B} &= sin\ A\ cos\ B - cos\ A\ sin\ B \end{aligned}\right\} \quad . \quad (73)$$

$$\left.\begin{aligned} cos\ \overline{A+B} &= cos\ A\ cos\ B - sin\ A\ sin\ B \\ cos\ \overline{A-B} &= cos\ A\ cos\ B + sin\ A\ sin\ B \end{aligned}\right\} \quad . \quad (74)$$

Adding the two expressions in (73) together we obtain

$$sin\ \overline{A+B} + sin\ \overline{A-B} = 2\ sin\ A\ cos\ B \quad . \quad (75)$$

and from (74)

$$cos\ \overline{A+B} + cos\ \overline{A-B} = 2\ cos\ A\ cos\ B. \quad . \quad (76)$$

Also putting $A = B$ in (73) and (74) we have

$$sin\ 2A = 2\ sin\ A\ cos\ A \quad . \quad . \quad (77)$$

$$\begin{aligned} cos\ 2A &= cos^2 A - sin^2 A \quad . \quad . \quad (78) \\ &= 2\ cos^2 A - 1 \\ &= 1 - 2\ sin^2 A \end{aligned}$$

or
$$\cos^2 \frac{A}{2} = \frac{1 + \cos A}{2}$$
$$\sin^2 \frac{A}{2} = \frac{1 - \cos A}{2}. \quad . \quad . \quad . \quad (79)$$

From (73) and (74) we have

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}. \quad . \quad (80)$$

$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}. \quad . \quad (81)$$

Hence
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}. \quad . \quad . \quad (82)$$

Again
$$\sin^2 A + \cos^2 A = 1. \quad . \quad . \quad (83)$$

Hence
$$\tan^2 A + 1 = \sec^2 A. \quad . \quad . \quad (84)$$

and
$$\cot^2 A + 1 = \operatorname{cosec}^2 A. \quad . \quad . \quad (85)$$

Important expressions for sine and cosine are

$$\sin A = \frac{\epsilon^{jA} - \epsilon^{-jA}}{2j} \quad . \quad . \quad . \quad (86)$$

$$\cos A = \frac{\epsilon^{jA} + \epsilon^{-jA}}{2}, \quad . \quad . \quad . \quad (87)$$

where $j = \sqrt{-1}$. Hence we have

$$\epsilon^{jA} = \cos A + j \sin A. \quad . \quad . \quad (88)$$

11. Differential Calculus

The differential calculus is a department of mathematics concerned with variable quantities or with rates of change of quantities. For instance, if a stone is falling freely downwards through the air, we may ask: (1) How far has it fallen since it started from rest? (2) At what rate is it moving at that instant, or what is its velocity? (3) What is the rate at which its speed or velocity is changing?

The answer to the first question is merely a length or distance. The answer to the second and third questions are rates, *i.e.* so many feet per second or so many feet per second per second. In the same manner if $y = f(x)$ is the ordinate of any curve, and if we give to x, the abscissa, a small *increment*

or increase we may ask, How does the increment of the ordinate y vary when x varies? The answer to this question is given by the *differential co-efficient* or *differential* of y with respect to x.

Thus, let x become $x+h$, then $f(x)$ becomes $f(x+h)$ and $f(x+h)-f(x)$ is called the *increment of the function* $f(x)$. Also h is called the *increment of the variable* x. Now the ratio of $f(x+h)-f(x)$ to h approaches a limit as h decreases without limit, and this is written, $\frac{dy}{dx}$ = the limiting value of $\left(\frac{f(x+h)-f(x)}{h}\right)$, as h diminishes without limit = the differential of $f(x)$ with respect to x.

Thus if the function is $y=x^2=f(x)$, then $f(x+h=(x+h)^2$ $=x^2+2hx+h^2$ and $\frac{f(x+h)-f(x)}{h}=2x+h$.

If h diminishes without limit the above expression $2x+h$ becomes $2x$. Hence if $y=x^2$ we have

$$\frac{dy}{dx}=2x. \quad . \quad . \quad . \quad (89)$$

The differential of x^2 with respect to x is therefore $2x$.

In the same manner we can show by the aid of the binomial theorem that if $y=x^n$, $\frac{dy}{dx}=nx^{n-1}$.

The process of obtaining $\frac{dy}{dx}$ from y is called *differentiating* y with regard to x.

Again, take the function $y=sin\ x=f(x)$.

Then $f(x+h)=sin\ (x+h)=sin\ x\ cos\ h+cos\ x\ sin\ h$ and $\frac{f(x+h)-f(x)}{h}=cos\ x\frac{sin\ h}{h}-sin\ x\frac{1-cos\ h}{h}$.

Now as h continually diminishes without limit, $sin\ h$ approaches h in value and $\frac{sin\ h}{h}$ becomes equal to 1. Also since $1-cos\ h=2\ sin^2\frac{h}{2}$, we can write instead of $\frac{1-cos\ h}{h}$, $\frac{sin\frac{h}{2}}{\frac{h}{2}}sin\frac{h}{2}$, and hence $\frac{1-cos\ h}{h}$ continually approaches zero as h decreases without limit.

Hence the limit of the expression $\frac{f(x+h)-f(x)}{h}$ is $\cos x$, or $\frac{dy}{dx}=\cos x$.

We can in the above manner find the differential coefficients of a number of simple algebraic or trigonometrical functions.

In some cases they can be derived from each other as follows :—

Let $y=\sin^{-1}x$. Then $x=\sin y$.

Hence $\frac{dx}{dy}=\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-x^2}$.

Therefore $\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$.

The following table contains some of the most frequently required fundamental, differential coefficients :—

Function.	*Differential Coefficient.*
1. $y=x^n$	$\frac{dy}{dx}=n\,x^{n-1}$
2. $y=\sqrt{x}=x^{\frac{1}{2}}$	$\frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$
3. $y=\sin x$	$\frac{dy}{dx}=\cos x$
4. $y=\cos x$	$\frac{dy}{dx}=-\sin x$
5. $y=\tan x$	$\frac{dy}{dx}=\frac{1}{\cos^2 x}=\sec^2 x$
6. $y=\sin^{-1}x$	$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$
7. $y=\cos^{-1}x$	$\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}}$
8. $y=\tan^{-1}x$	$\frac{dy}{dx}=\frac{1}{1+x^2}$
9. $y=\log_e x$	$\frac{dy}{dx}=\frac{1}{x}$

The following rules are important.

Rule 1.—To differentiate the product of two functions, differentiate each function separately, and multiply by the other ; then add these products.

Thus if $y=AB$ where A and B are each functions of x, we have

$$\frac{dy}{dx}=A\frac{dB}{dx}+B\frac{dA}{dx}. \quad . \quad . \quad (90)$$

Rule 2.—To differentiate the quotient of two functions, differentiate the numerator and multiply it by the denominator. Also differentiate the denominator and multiply it by the numerator. Subtract the results and divide by the square of the denominator.

Thus if $y = \frac{A}{B}$ where A and B are both functions of x, we have

$$\frac{dy}{dx} = \frac{B\frac{dA}{dx} - A\frac{dB}{dx}}{B^2}. \quad . \quad . \quad . \quad (91)$$

Rule 3.—To differentiate an exponential such as a^A where a is a constant and A is a function of x, multiply together the exponential, the Napierian logarithm of the base, and the differential of the exponent with respect to the variable. Thus if we have $y = a^A$, then

$$\frac{dy}{dx} = log_e a . a^A \frac{dA}{dx}. \quad . \quad . \quad . \quad (92)$$

Examples.—If $y = e^x$, then

$$\frac{dy}{dx} = e^x. \quad . \quad . \quad . \quad (93)$$

If $y = a^x$, then

$$\frac{dy}{dx} = log_e a . a^x. \quad . \quad . \quad . \quad (94)$$

If $y = a^{x^2}$, then

$$\frac{dy}{dx} = a^{x^2} log_e a . 2x. \quad . \quad . \quad (95)$$

Rule 4.—If y is a function of z and z is a function of x, then to find $\frac{dy}{dx}$ we find first $\frac{dy}{dz}$ and then multiply by $\frac{dz}{dx}$.

Thus

$$\frac{dy}{dx} = \frac{dy}{dz} . \frac{dz}{dx}. \quad . \quad . \quad . \quad (96)$$

Example.—Let $y = sin\ ax^2$, then $\frac{dy}{dx} = 2ax\ cos . ax^2$.

Let $y = log_e (ax^2 + bx)$, then $\frac{dy}{dx} = \frac{2ax + b}{ax^2 + bx}$.

Since $y = f(x)$ represents the ordinate of some curve corresponding to some abscissa x, it is clear that $y = f(x + h)$ represents the ordinate of that curve corresponding to an

abscissa $x+h$. Hence the differential coefficient $\frac{dy}{dx}$ denotes the slope of the curve, or what is the same thing, the tangent of the angle of inclination to the axis of x of a straight line drawn to touch the curve or be a tangent to it at that point.

If, therefore, the curve has a horizontal tangent at any point, then corresponding to the abscissa of that point the ordinate y has either a maximum or a minimum value, and the differential coefficient $\frac{dy}{dx}$ is zero at that point.

Hence if we have any function of x which represents some curve, and if we find the value of the differential coefficient of the function with respect to x, and if we equate this to zero, it will give us an expression from which we can determine the values of the abscissa for which the ordinate becomes either a maximum or a minimum.

Thus if $y=sin\frac{2\pi}{\lambda}x$, we have seen that this represents a simple harmonic or simple period wave-like curve.

Now
$$\frac{dy}{dx}=\frac{2\pi}{\lambda}cos\frac{2\pi}{\lambda}x.$$

This last function has zero values when

$$x=\frac{\lambda}{4},\quad \frac{3\lambda}{4},\quad \frac{5\lambda}{4},\quad \text{etc.}$$

Hence corresponding to these values of x we have $y=+1$ or -1, viz. a maximum positive or negative.

An important theorem with regard to the ordinates of a curve is as follows:

If $y=f(x)$ represents the ordinate of any curve corresponding to an abscissa x, then $y+\frac{dy}{dx}\delta x$ (see Fig. 5) represents the ordinate of that curve corresponding to an abscissa $x+\delta x$ where δx is a small increment of x. For the slope of the curve at the abscissa x is $\frac{dy}{dx}$. This denotes the trigonometrical tangent of the angle which the geometrical

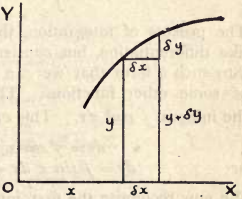


FIG. 5.

tangent line at that point makes with the axis of x. Hence the increase in the ordinate corresponding to the increment δx of the abscissa must be $\frac{dy}{dx}\delta x$, and therefore the total ordinate must be $y + \frac{dy}{dx}\delta x$ corresponding to $x + \delta x$.

12. Integral Calculus

Integration is the inverse operation to differentiation. In the case of the latter, the function is given and it is required to find the rate at which it changes with a change in the variable. In the case of integration the rate of change of the function or the differential coefficient is given, and it is required to find the function from which it was derived. Hence integration is for the most part a guess-work process depending on our knowledge of the differential coefficients yielded by certain types of function. Thus, for instance, if we are given the differential coefficient $\frac{dy}{dx} = 2x$ we can see at once that its integral is $y = x^2 + C$ where C is some constant. If we are given any power of a variable where the exponent is a number such as x^n, then the integral is obtained by raising the exponent one unit and dividing by the raised exponent and adding a constant. Thus if we are given $\frac{dy}{dx} = x^n$ we have $y = \frac{x^{n+1}}{n+1} + C$. The symbol for integration is a long S or $\int$, and the above result is written

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C. \quad . \quad . \quad (97)$$

The process of integration, therefore, is not a direct process like differentiation, but consists in putting the given function into such a form that we can recognise it as the differential of some other function. Thus let it be required to find the integral $\int sin^3 dx$. This can be put in the form

$$dy = \int sin\, x\, (1 - cos^2 x)\, dx$$

or

$$dy = \int sin\, x\, dx - \int cos^2 x\, sin\, x\, dx.$$

We now recognise the first integral to be $-cos\, x$ because this last function differentiated with regard to x gives $sin\, x$. Also the second integral is recognised to be $\frac{1}{3} cos^3 x$ because this differentiated with regard to x gives $-cos^2 x\, sin\, x$.

Hence we have

$$y = -\cos x + \tfrac{1}{3}\cos^3 x + C \quad . \quad . \quad (98)$$

as the value of $\int \sin^3 x dx$.

An integral is called a definite integral when it is taken between certain values of the variable. This is denoted by small letters put at the top and bottom of the integration sign. Thus

$$\int_0^{2\pi} \sin^3 x dx$$

means that after we have found the general integral we have to substitute for the variable successively the upper limit and the lower limit, and then subtract the latter value from the former value. Thus if we have the definite integral $\int_0^{2\pi} \sin^3 x dx$ we have the general integral as already found to be $y = -\cos x + \frac{1}{3}\cos^3 x + C$. Substitute for x the value 2π for which $\cos x = -1$ and we obtain $1 - \frac{1}{3} = \frac{2}{3}$. Next substitute for x the value 0 for which $\cos x = 1$ and we have $-1 + \frac{1}{3} = -\frac{2}{3}$. Subtracting the latter from the former we have $\frac{4}{3}$ which is the value of $\int_0^{2\pi} \sin^3 x dx$.

The following is a table of certain often-required integrals :—

Table of Integrals often required

Note.—C is the constant of integration and the logarithms are Napierian.

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C.$
2. $\int \frac{dx}{x} = \log_e x + C.$
3. $\int e^x dx = e^x + C.$
4. $\int \sin x dx = -\cos x + C.$
5. $\int \cos x dx = \sin x + C.$
6. $\int \tan x dx = -\frac{1}{2}\log_e(\cos^2 x) + C.$
7. $\int \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1} x + C.$
8. $\int \frac{dx}{\sqrt{x^2 \pm 1}} = \log\left\{x + \sqrt{x^2 \pm 1}\right\} + C.$
9. $\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = \frac{1}{a}\log\left\{\frac{x}{a + \sqrt{a^2 \pm x^2}}\right\} + C.$

10. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = log\left\{x + \sqrt{x^2 \pm a^2}\right\} + C.$

11. $\int e^{mx} \cos mx dx = e^{mx}\left(\frac{n \cos mx + m \sin mx}{m^2 + n^2}\right) + C.$

12. $\int e^{mx} \sin mx dx = e^{mx}\left(\frac{n \sin mx - m \cos mx}{m^2 + n^2}\right) + C.$

13. $\int \sqrt{x^2 - a^2} dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} log_e\left\{x + \sqrt{x^2 - a^2}\right\} + C.$

14. $\int \frac{dx}{\sqrt{x^2 - ax}} = log_e\left(x - \frac{a}{2} + \sqrt{x^2 - ax}\right) + C.$

13. Determination of Lengths of Curves, and Areas, and Volumes by Integration

If y is the ordinate of any curve, and x the corresponding abscissa, and if another ordinate of length $y + \delta y$ is drawn corresponding to an abscissa $x + \delta x$, then the area contained between the two ordinates, the curve and the axis of x, becomes nearly equal to $y\delta x$, as δx becomes smaller. In the limit, therefore, we may call ydx the element of area dA, and the area included by the curve between two ordinates corresponding to abscissæ $x = a$ and $x = b$, and the axis of x, is given by the definite integral

$$A = \int_a^b ydx. \quad . \quad . \quad . \quad (99)$$

Hence if y is given as a function of x we can often evaluate this integral and obtain an expression for the area.

Example.—Let the curve be a parabola. Then if we take the origin at the vertex, the equation to the parabola is $y = 2\sqrt{a}\sqrt{x}$ where a is some constant. Hence the area included by the curve, the axis of x and an ordinate y, is given by

$$A = \int_0^x 2\sqrt{a}\sqrt{x}dx$$

$$= \frac{4\sqrt{a}}{3}x^{\frac{3}{2}} = \frac{2}{3}yx.$$

Hence the area of a parabola is two-thirds of that of the circumscribing rectangle.

Example.—Find the area of a circle. Taking the centre as origin we draw two radii separated by a small angle $d\theta$.

Let r be the radius of the circle. Then the area of this narrow sector is the area of a triangle of height r and base equal to $rd\theta$, which is the length of the short arc forming the base. Hence the element of area is $dA = \frac{1}{2}r^2 d\theta$, and the whole area of the circle is

$$A = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \pi r^2.$$

In the same manner we can find the volume of a solid by integrating the areas of elementary slices.

Example.—Find the area of a right circular cone, that is, a cone with circular base. Take the vertex of the cone as origin and consider a normal circular slice of it taken at a distance x from the vertex, and of thickness dx. Then the radius of this circular slice is $x \tan\theta$, where 2θ is the inclination of the generating or bounding lines of the cone. Hence the volume of the slice is $\pi x^2 \tan^2\theta dx$.

The whole volume of the cone is

$$\int_0^x \pi x^2 \tan^2\theta dx = \frac{1}{3}\pi \tan^2\theta x^3$$

$$= \frac{x}{3}\pi x^2 \tan^2\theta.$$

Therefore the volume of a cone is equal to one-third of the height multiplied by the base, or to one-third of the circumscribing cylinder.

The following rules for calculation of areas and volumes are often needed.

The ratio of the circumference of a circle to its diameter is denoted by the Greek letter $\pi = 3.14159265 = \frac{22}{3}$ roughly. Then

The area of

a triangle $= \frac{1}{2}$ base $\times$ height $= \frac{bh}{2}$.

a circle $= \pi \times$ radius squared $= \pi r^2$.

an ellipse $= \pi \times$ product of semi-major axis and semi-minor axis $= \pi ab$.

a parabola $= \frac{2}{3}$ circumscribing rectangle.

The volume of

a right circular cone $= \frac{1}{3}$ area of base $\times$ height.

a sphere $= \frac{4}{3} \times \pi \times$ radius cubed $= \frac{4}{3}\pi r^3$.

The surface of

a sphere $= 4 \times \pi \times$ radius squared $= 4\pi r^2$.

14. Second and Higher Differentials: Expansion in Series

We can not only differentiate a function of a variable, but also the differentials themselves. Thus $\frac{dy}{dx}$ denotes the rate at which y changes with x when y is a function of x.

In the same way we can consider the change in $\frac{dy}{dx}$ with x. This is denoted by $\frac{d^2y}{dx^2}$, and is called the second differential coefficient of y. Also we can write $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, $\frac{d^ny}{dx^n}$ as expressions for the 3rd, 4th, and nth differential coefficients of y.

An important theorem is that of Maclaurin, which shows us how to express certain functions in a series of terms involving the various differential coefficients of that function.

Let $y = f(x)$ be any function of x, and let $f_1(x)$, $f_2(x)$, $f_3(x)$ be the successive differential coefficients of y with regard to x. Then let it be assumed that $f(x)$ can be expanded into a series of terms involving ascending powers of x of the form

$$f(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + \text{etc.} \quad . \quad (100)$$

Differentiate $f(x)$ successively, and then after differentiation put $x = 0$ in each expression. We have

$$f_1(x) = A_1 + 2A_2x + 3A_3x^2 + \text{etc.}$$
$$f_2(x) = 2A_2 + 2.3A_3x + 3.4A_4x^2 + \text{etc.}$$
$$f_3(x) = 2.3A_3 + 2.3.4x + \text{etc.}$$

If then we put $x = 0$ in the above series and denote by $\left(\frac{dy}{dx}\right)$, $\left(\frac{d^2y}{dx^2}\right)$, $\left(\frac{a^3y}{dx^3}\right)$ the values the successive differential coefficients then assume when $x = 0$, and if we put $f(x_0)$ for the

value assumed by the function $f(x)$ when $x = 0$, we see that we can express $f(x)$ as follows :—

$$f(x) = f(x_0) + \left(\frac{dy}{dx}\right)x + \frac{1}{2!}\left(\frac{d^2y}{dx^2}\right)x + \frac{1}{3!}\left(\frac{d^3y}{dx^3}\right)x^3 + \text{etc.}, \quad . \quad (101)$$

where $2! = 1 \times 2$, $3! = 1 \times 2 \times 3$, $4! = 1 \times 2 \times 3 \times 4$ etc.

The above expansion is called Maclaurin's Theorem. Its use can best be illustrated by an example or two.

Example.—Expand ϵ^x in a series of powers of x. We see that $\frac{d}{dx}(\epsilon^x) = \epsilon^x$, and hence all the successive differential coefficients of ϵ^x are also equal to ϵ^x. Also we see that when $x = 0$ the function ϵ^x and all its differential coefficients become unity. Hence we have

$$\epsilon^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \text{etc.} \quad . \quad . \quad (102)$$

This is the exponential theorem already given.

Example 2.—Expand $y = \sin x$ in a series of powers of x. We have

$$\frac{dy}{dx} = \cos x, \quad \frac{d^2y}{dx^2} = -\sin x, \quad \frac{d^3y}{dx^3} = -\cos x, \text{ etc.}$$

When $x = 0$ these functions take the values 0, 1, 0, -1, etc., respectively. Hence we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \text{etc.} \quad . \quad . \quad (103)$$

Also in the same way we can find

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \text{etc.} \quad . \quad . \quad (104)$$

If we expand in the same manner the function ϵ^{jx}, where $j = \sqrt{-1}$, it is easy to prove that

$$\epsilon^{jx} = \cos x + j \sin x, \quad . \quad . \quad (105)$$

and

$$\epsilon^{-jx} = \cos x - j \sin x; \quad . \quad . \quad (106)$$

and hence that

$$\sin x = \frac{\epsilon^{jx} - \epsilon^{-jx}}{2j}, \quad . \quad . \quad (107)$$

$$\cos x = \frac{\epsilon^{jx} + \epsilon^{-jx}}{2}. \quad . \quad . \quad (108)$$

These formulæ are called the exponential values of sine and cosine.

15. Differential Equations

A differential equation is a mathematical statement expressing some equality between quantities, some of which are differential coefficients and others are functions of the variables and others are constants.

Thus a simple form of differential equation is

$$\frac{dy}{dx} + ay = 0. \quad . \quad . \quad . \quad (109)$$

Here we have some quantity y which is a function of x, the variable, and the equation tells us that there is a relation between y, $\frac{dy}{dx}$, and a constant a expressed as above. The solution of a differential equation implies finding some function y which satisfies this relation as expressed. If then we write the equation (109) in the form

$$\frac{dy}{y} = -adx$$

we can easily see that the left-hand side is the differential of $log_e y$ and the right-hand side is the differential of $-ax$. Hence, integrating each side we have

$$C + log_e y = -ax,$$

where C is some constant of integration. Let us denote this constant by $-log_e A$.

Then we have

$$log_e \frac{y}{A} = -ax,$$

or

$$y = A\epsilon^{-ax}, \quad . \quad . \quad . \quad (110)$$

This last expression is therefore a solution of (109), as can easily be seen by differentiating it with respect to x.

A differential equation may involve second or higher-differential coefficients. Thus, for instance, let the equation be

$$\frac{d^2y}{dx^2} + a^2y = 0. \quad . \quad . \quad . \quad (111)$$

An equation of the above type is called a *linear* differential equation, and it can always be solved by the assumption that y is a function of x of the form $y = \epsilon^{mx}$, and all that remains is to find the value of m.

If we substitute the above value of y in the original equation (111) we obtain

$$m^2y = -a^2y.$$

Hence we have $m = ja$, where $j = \sqrt{-1}$.

Therefore a particular solution of (111) is

$$y = A\epsilon^{jax},$$

where A is some constant.

Since $\epsilon^{jax} = \cos ax + j \sin ax$ we can say that other solutions of (111) are $y = A \cos ax$, and $y = Aj \sin ax$ or a complete solution of (111) is

$$y = A \cos ax + B \sin ax. \quad . \quad . \quad (112)$$

This shows us that a differential equation may have several solutions, and we have to distinguish between a general solution and a particular solution.

In the same manner we can show that a solution of the equation

$$\frac{d^2y}{dx^2} - a^2y = 0$$

is

$$y = A\epsilon^{-ax} + B\epsilon^{+ax} \quad (113)$$

The values of the constants A and B have to be determined from the limiting conditions of the problem, such as the value of y when x is zero or infinity, or the value of $\frac{dy}{dx}$ when x is zero or infinity.

A type of differential equation often occurring in physical investigations is

$$A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = 0, \quad . \quad . \quad (114)$$

where A, B, and C are constants.

To solve this, assume $y = \epsilon^{mx}$ and substitute this value for y in the original equation.

We obtain after dividing all through by ϵ^{mx} a quadratic equation, viz. :—

$$Am^2 + Bm + C = 0,$$

or

$$m^2 + 2\frac{B}{2A}m = -\frac{C}{A}.$$

Solving this quadratic equation in m we have

$$m + \frac{B}{2A} = \pm \frac{\sqrt{B^2 - 4AC}}{2A},$$

$$m = -\frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A} \quad . \quad . \quad (115)$$

Suppose that $4AC$ is greater than B^2, then the above may be written

$$m = \alpha + j\beta, \quad . \quad . \quad . \quad (116)$$

where $$\alpha = -\frac{B}{2A} \text{ and } \beta = \pm \frac{\sqrt{4AC - B^2}}{2A} \quad . \quad (117)$$

Hence the solution of (114) is

$$y = \epsilon^{mx} = \epsilon^{(\alpha+j\beta)x} = \epsilon^{\alpha x}(\cos \beta x + j \sin \beta x),$$

or $$y = \epsilon^{\alpha x}(A \cos \beta x + B \sin \beta x). \quad . \quad (118)$$

If, however, B^2 is greater than $4AC$, then the solution will be of the form

$$y = \epsilon^{(\alpha+\beta)x}. \quad . \quad . \quad . \quad (119)$$

The graphical representation of this last solution (119) is an exponential curve, whereas the graphical representation of the solution (118) is a decrescent periodic curve or a wavy line which gradually decreases in amplitude.

Another typical differential equation is

$$\frac{dy}{dx} + ay = b. \quad . \quad . \quad . \quad (120)$$

This differs from (109) only in having a constant b on the right-hand side in place of a zero.

The solution of (120) is

$$y = C\epsilon^{-ax} + \frac{b}{a}, \quad . \quad . \quad (121)$$

where C is some constant, such that $C + \frac{b}{a}$ is the value of y when $x = 0$.

It is easy to verify this solution by differentiating (121) with respect to x and adding it to the product of ay, when it will be found to be equal to b.

A more general form of this equation is

$$\frac{dy}{dx} + Py = Q, \quad . \quad . \quad . \quad (122)$$

where P and Q are both functions of x. It can be solved by the aid of an *integrating factor*. For if we multiply all through by $\epsilon^{\int Pdx}$ we obtain

$$\epsilon^{\int Pdx}\frac{dy}{dx} + Py\epsilon^{\int Pdx} = Q\epsilon^{\int Pdx}. \quad . \quad . \quad (123)$$

Now $P = \frac{d}{dx}\int Pdx$ since the operation $\int(\ )dx$ is the inverse

of $\frac{d}{dx}$. Hence we see, then, that the left-hand side of equation (123) is the complete differential of the product $y\epsilon^{fPdx}$.

Hence we can say that

$$y\epsilon^{fPdx} = \int Q\epsilon^{fPdx}dx + c,$$

where c is some constant of integration.

Therefore the solution of (122) is

$$y = \epsilon^{-fPdx}\{\int Q\epsilon^{fPdx}dx + c\}. \quad . \quad . \quad (124)$$

For example, solve the equation

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}\sin pt, \quad . \quad . \quad (125)$$

where t is the variable and R, L, E and p are constants.

Hence $\frac{R}{L} = P$ and $Q = \frac{E}{L}\sin pt$.

Therefore $i = \epsilon^{-\frac{R}{L}t}\left(\frac{E}{L}\int \epsilon^{\frac{R}{L}t}\sin pt dt + c\right)$.

Now $\int \epsilon^{\frac{R}{L}t}\sin pt = \frac{\epsilon^{\frac{R}{L}t}}{R^2 + p^2L^2}\left(LR\sin pt - pL^2\cos pt\right)$

(see previous Table of Integrals, § 12); and hence we have

$$i = \frac{E}{\sqrt{R^2 + p^2L^2}}\sin\overline{pt - \theta} + \epsilon^{-\frac{R}{L}t}. \quad (126)$$

as the solution of (125). In the above equation

$$\theta = \tan^{-1}\frac{Lp}{R}.$$

The reader should bear in mind that

$$A\sin x + B\cos x = \sqrt{A^2 + B^2}\sin\overline{x + \theta} \quad . \quad (127)$$

where $\tan\theta = \frac{B}{A}$.

This is easily proved. Since if $\tan\theta = \frac{B}{A}$ we have

$$\sin\theta = \frac{B}{\sqrt{A^2 + B^2}} \quad \text{and} \quad \cos\theta = \frac{A}{\sqrt{A^2 + B^2}},$$

and since $\sin\overline{x + \theta} = \sin x\cos\theta + \cos x\sin\theta$, the equation (127) follows at once.

Finally, we often meet with a differential equation of the form

$$P\frac{d^ny}{dx^n} + Q\frac{d^{n-1}y}{dx^{n-1}} \ldots + U\frac{dy}{dx} + Wy = 0, \quad . \quad (128)$$

which is called a *linear differential equation* of the first degree, because the various differential coefficients enter only in the first power. We can always solve this differential equation as follows :—

$$\text{Let } y = \epsilon^{mx}, \text{ then } \frac{d^n y}{dx^n} = m^n y.$$

Substituting, therefore, the values of the various differential coefficients, and dividing all through by ϵ^{mx} we have an algebraic equation of the nth degree, viz. :—

$$Pm^n + Qm^{n-1} + \ldots Um + W = 0.$$

This equation has n roots or values of m which satisfy it. If these can be determined, then there will be some real roots differing from one another, some imaginary or complex roots of the form $\alpha + j\beta$, $\alpha - j\beta$ which always occur in pairs, and there may be some pairs of identical roots or equal roots. Then we have the following rules.

I. If the roots are all real, let them be denoted by m_1, m_2, m_3, etc.

Then the solution of (128) is

$$y = A\epsilon^{m_1 x} + B\epsilon^{m_2 x} + C\epsilon^{m_3 x} + \text{etc.}, \quad . \quad (129)$$

and the constants A, B, C, must be determined from the limiting conditions of the problem, that is, from the values of the functions y or $\frac{dy}{dx}$, etc., when $x = 0$ or $x =$ infinity.

II. If the roots are complex, that is of the form $\alpha \pm j\beta$, then the solution is of the form

$$y = \epsilon^{\alpha x}(A \cos \beta x \pm B \sin \beta x), \quad . \quad (130)$$

and one such pair of terms will occur for every different pair of imaginary roots.

III. If there are a pair of equal real roots, say m_1 and m_1, then the solution is of the form

$$y = (A + Bx)\,\epsilon^{m_1 x}, \quad . \quad . \quad . \quad (131)$$

and if there are three real equal roots it is of the form

$$y = (A + Bx + Cx^2)\,\epsilon^{m_1 x}. \quad . \quad . \quad (132)$$

For example, suppose the equation given to be

$$\frac{d^5 y}{dx^5} - \frac{d^3 y}{dx^3} - 2\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} = 0. \quad . \quad . \quad (133)$$

Then the auxiliary equation is,

$$m^5 - m^3 - 2m^2 + 2m = 0.$$

It is easily seen that this breaks up into the factors

$$m(m-1)(m-1)(m^2+2m+2)=0;$$

and the solution of $m^2+2m+2=0$

is $$m=-1\pm\sqrt{-1}.$$

Hence the five roots are

$$m=0,\ 1,\ 1,\ (-1+\sqrt{-1}),\ (-1-\sqrt{-1}),$$

and the solution of (133) is

$$y=A+(B+Cx)e^{x}+De^{-x}\cos x+Ee^{-x}\sin x. \quad (134)$$

If the right-hand side of a linear differential equation is not zero but a constant or a function of x then special methods are necessary for its solution, and the reader must be referred to some good treatise such as Boole's *Differential Equations.*

Another frequently occurring type of differential equation is

$$c^2\frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}. \quad . \quad . \quad . \quad (135)$$

The solution of this is

$$y=f(x-ct)+f(x-ct), \quad . \quad . \quad (136)$$

where $f(x)$ stands for some function of x and $f(x-ct)$ or $f(x+ct)$ for the same function of $x-ct$ or $x+ct$.

A function $y=f(x)$ stands for, or denotes some curve *at rest* whose ordinate is y at an abscissa x. The function $f(x-ct)$ stands for the same curve *in motion* with a velocity c. For $f(x-ct)$ retains the same value if x is increased by x' and t by t', provided

$$\frac{x'}{t'}=c.$$

Hence (128) denotes a wave motion or curve in motion with velocity c.

Another important type of differential equation is

$$\frac{d^2y}{dr^2}+\frac{1}{r}\frac{dy}{dr}+m^2y=0. \quad . \quad (137)$$

This is called Bessel's Equation. It cannot be solved in finite terms, but has a solution in the form of a series denoted by $J_0(mr)$ when

$$y=J_0(mr)=1-\frac{(mr)^2}{2^2}+\frac{(mr)^4}{2^2 4^2}-\frac{(mr)^6}{2^2 4^2 6^2}+\text{etc.} \quad (138)$$

Tables of the values of $J_0(mr)$ have been calculated for various values of mr.

16. Vector Analysis

We have already explained the methods of denoting a vector quantity and the meaning of the terms scalar and vector multiplication. These two forms of multiplication are respectively denoted by a *dot* and by a *cross* placed between letters or symbols in thick type which denote the vectors. We can represent a vector as the vector sum of three component vectors directed along three axes *Ox*, *Oy*, *Oz*, at right angles to each other. Thus if **r** is such a vector and **i**, **j**, **k** are unit vectors along the axes *x*, *y*, and *z*, we have

$$\mathbf{r} = \mathbf{i}r_1 + \mathbf{j}r_2 + \mathbf{k}r_3, \qquad . \qquad . \quad (139)$$

where r_1 r_2 r_3 are the magnitudes or sizes of the component vectors along *x*, *y*, and *z*.

We have frequent occasion to use an operator denoted by the symbol ∇ called the *del* operator which is equivalent to

$$\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \nabla. \qquad . \qquad . \quad (140)$$

This operator tells us to add together vectorially the operations or results of differentiating some expression or function with regard to *x*, *y*, and *z* respectively, along the three axes of *x*, *y*, and *z*. The effect of this operation depends on whether the *del* operator acts as a scalar quantity or on a vector quantity.

Definition 1.—Any region in which a scalar quantity is distributed so that its value is some function of the co-ordinates is called a *scalar field* and the quantity a *scalar point function.*

Definition 2.—Any region in which a vector quantity is distributed so that its value is some function of its co-ordinates is called a *vector field* and a *vector point function.*

For example, a map which shows the distribution of temperature at any place simultaneously in the British Isles is a scalar field, because *temperature* is a scalar quantity; but a map which shows the wind velocity everywhere at any time is a vector field, because *velocity* is a vector quantity.

Suppose then that we have a scalar field we can unite all points at which this scalar has the same values by a line called a *level line* or *equipotential line* or *isothermal line* according as the scalar quantity considered is height, potential or temperature.

If we draw two such lines very near to each other or differing very little in scalar value, and if we select a point on one line and a point on the other and join these two points by a

straight line, there will be a certain change in the value of the scalar quantity in moving down this last straight line the rate of which is called the *slope* of the scalar. Thus to fix ideas let the lines be equipotential lines drawn in an electric field and connecting all points at which the potential V has the same value.

Definition 1.—The potential V at any point in an electric field is measured by the work done in bringing up a small body charged with a unit of positive electricity to that point from an infinite distance.

Definition 2.—Equipotential lines are lines along which there is no change in potential.

The electric force E acting at any point in an electric field in a direction x is measured by the rate of decrease of potential in that direction or by

$$E = -\frac{dV}{dx}. \quad . \quad . \quad . \quad (141)$$

The electric force at any point is normal or perpendicular to the equipotential line passing through that point. For if it were not, then the electric force would have a resolved part or component along the equipotential line, and this would involve work in moving electricity along that line and it would not then be an equipotential line.

Definition 3.—A vector drawn perpendicular to an equipotential line at any point, the size being proportional to the fastest rate of increase of the potential V at that point, and drawn in the direction of this fastest increase, is called the *gradient* of the potential at that point and denoted by *Grad V*.

Hence the electric force at any point is equal to $-$ *Grad V* at that point.

Suppose any straight line drawn in the electric field extending from a point where the potential is V to a point where it is $V + dV$. Let the length of this line be $\mathbf{dr}$. Then the work done in moving a unit of positive electricity along this line is $\mathbf{E} \cdot \mathbf{dr}$, because it is the scalar product of the vector $\mathbf{dr}$ and the vector representing the electric force $\mathbf{E}$. Now $\mathbf{E} = -$ *Grad V*, and therefore $\mathbf{E} \cdot \mathbf{dr} = -$ *Grad* $\mathbf{V} \cdot \mathbf{dr}$. But this work must be equal to dV. Now if V is a function of the co-ordinates x, y, and z of the point, then

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{dy}dy + \frac{\partial V}{\partial z}dz \quad . \quad . \quad (142)$$

and the above is equal to the scalar product

$$\left(\mathbf{i}\frac{\partial V}{\partial x}+\mathbf{j}\frac{\partial V}{\partial y}+\mathbf{k}\frac{\partial V}{\partial z}\right)\cdot\left(\mathbf{i}dx+\mathbf{j}dy+\mathbf{k}dz\right), \quad . \quad (143)$$

where **i**, **j**, and **k** are unit vectors along the axes. The contents of the first brackets are denoted by ∇V or the delta operator acting on V, and that of the second brackets is **dr**. Hence we have an equation

$$Grad\ V\cdot\mathbf{dr}=\nabla V\cdot\mathbf{dr}$$

or

$$\nabla V=Grad\ V. \quad . \quad . \quad . \quad (144)$$

Hence ∇ or the *del* operator applied to a scalar point function V gives us the *gradient* of V or the fastest space rate of change of V. (See J. G. Coffin, *Vector Analysis*, Chapman, Hall & Co., London, p. 103.)

Let us consider next the meaning of $\nabla\cdot\mathbf{F}$ or the scalar product of ∇ and a vector **F**.

We may express **F** as the sum of its components along the three axes of x, y, and z or

$$\mathbf{F}=\mathbf{i}F_1+\mathbf{j}F_2+\mathbf{k}F_3 \quad . \quad . \quad (145)$$

and

$$\nabla=\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}. \quad . \quad . \quad (146)$$

Hence by formula (43) and the rules for scalar multiplication we have

$$\nabla\cdot\mathbf{F}=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}. \quad . \quad . \quad (147)$$

The operator $\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}$ is called *Div*, short for *divergence*, and the function (147) is called the *divergence* of **F**. It means the amount or quantity of **F** which originates in, or diverges from, a unit volume of space.

This is proved as follows. Consider a small rectangular volume whose sides are dx, dy, dz in length. Then let F_1 be the value of the x-component of the vector at x, then $F_1+\frac{dF_1}{dx}dx$ will be its value at $x+dx$ and the increment or gain in F_1 will be $\frac{dF_1}{dx}dx$ in the length dx. If F_1 is the value per unit area then the quantity or amount or surface integral of F_1 gained in the length dx will be $\frac{dF_1}{dx}dx.dy.dz$. The same for the flows or gains in the other axial directions. Hence the total gain in the volume $dx.dy.dz$ will be the

sums of $\frac{dF_1}{dx}$, $\frac{dF_2}{dy}$, $\frac{dF_3}{dz}$, each multiplied by $dx.dy.dz$. Hence the increment or divergence per unit volume is

$$\nabla \cdot \mathbf{F} = \frac{dF_1}{dx} + \frac{dF_2}{dy} + \frac{dF_3}{dz} = Div\ \mathbf{F}. \quad . \quad (148)$$

If this sum is zero the vector is said to have no divergence. If a vector is distributed through a field so that it is represented as to direction by endless lines, such as is the case with magnetic flux and electric current, it is a *non-divergent vector.* If, however, it is represented by lines which start from some points, as is the case with magnetic force and electric force which originates at magnetic poles and electrified bodies, then the vector is a *divergent vector.*

We have next to consider the meaning of $\nabla \times \mathbf{F}$ or the vector product of the *del* operator and a vector $\mathbf{F}$; we have

$$\left.\begin{aligned} \mathbf{F} &= \mathbf{i}dF_1 + \mathbf{j}dF_2 + \mathbf{k}dF_3 \\ \text{and} \qquad \nabla &= \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \end{aligned}\right\} . \quad . \quad (149)$$

Hence by the rules of vector multiplication

$$\begin{aligned} \nabla \times \mathbf{F} &= \mathbf{i}\left(\frac{dF_3}{dy} - \frac{dF_2}{dz}\right) + \mathbf{j}\left(\frac{dF_1}{dz} - \frac{dF_3}{dx}\right) + \mathbf{k}\left(\frac{dF_2}{dx} - \frac{dF_1}{dy}\right) \\ &= \mathbf{i}C_1 + \mathbf{j}C_2 + \mathbf{k}C_3. \quad . \quad . \quad . \quad . \quad . \quad (150) \end{aligned}$$

The vector $\mathbf{C}$, whose components

$$\text{are } C_1 = \frac{dF_3}{dy} - \frac{dF_2}{dz},\ C_2 = \frac{dF_1}{dz} - \frac{dF_3}{dx},$$

$$\text{and } C_3 = \frac{dF_2}{dx} - \frac{dF_1}{dy}, \text{ is called the}$$

Curl of the vector $\mathbf{F}$.

The curl is itself a vector, and it is easily seen that since

$$\frac{dC_1}{dx} + \frac{dC_2}{dy} + \frac{dC_3}{dz} = 0 \quad . \quad . \quad (151)$$

the curl is always a non-divergent vector whatever the original vector $\mathbf{F}$ may be.

The expression for $\nabla \times \mathbf{F} = Curl\ \mathbf{F}$ may be written in determinant form

$$\nabla \times \mathbf{F} = Curl\ \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ F_1 & F_2 & F_3 \end{vmatrix} . \quad . \quad (152)$$

Since the curl of a vector is also a vector we can consider or calculate the curl of the curl of a vector. This is written *Curl*^{2} **F**, and clearly

$$Curl^2\ \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ C_1 & C_2 & C_3 \end{vmatrix}, \qquad . \quad (153)$$

where C_1, C_2, C_3 are the axial components of the curl, and have the values given above in terms of F_1, F_2, F_3.

Therefore

$$Curl^2\ \mathbf{F} = \mathbf{i}\left(\frac{dC_3}{dy} - \frac{dC_2}{dz}\right) + \mathbf{j}\left(\frac{dC_1}{dz} - \frac{dC_3}{dx}\right) + \mathbf{k}\left(\frac{dC_2}{dx} - \frac{dC_1}{dy}\right). \quad (154)$$

If we substitute in the above equation the values of C_1, C_2, and C_3 we obtain for the **i** or x-component of **C** the value

$$\frac{d^2F_2}{dxdy} - \frac{d^2F_1}{dy^2} - \frac{d^2F_1}{dz^2} + \frac{d^2F_3}{dxdz}, \qquad . \quad (155)$$

and if we add and subtract $\frac{d^2F_1}{dx^2}$ to (155) and rearrange terms (155) becomes

$$\frac{d}{dx}\left(\frac{dF_1}{dx} + \frac{dF_2}{dy} + \frac{dF_3}{dz}\right) - \left(\frac{d^2F_1}{dx^2} + \frac{d^2F_1}{dy^2} + \frac{d^2F_1}{dz^2}\right). \quad (156)$$

If then we carry out the same operation on the other terms of (154) it is easy to see we obtain the equation

$$Curl^2\ \mathbf{F} = \nabla\,(Div\ \mathbf{F}) - \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\mathbf{F}. \qquad (157)$$

The quantity in the last bracket is called Laplace's operator and written *Lap*. It is obvious that

$$Lap = \nabla \cdot \nabla = \nabla^2, \qquad . \quad . \quad (158)$$

because

$$Lap = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} = \left(\mathbf{i}\frac{d}{dx} + \mathbf{j}\frac{d}{dy} + \mathbf{k}\frac{d}{dz}\right) \cdot \left(\mathbf{i}\frac{d}{dx} + \mathbf{j}\frac{d}{dy} + \mathbf{k}\frac{d}{dz}\right).$$

In other words, *Lap* = the scalar product of *del* and *del*. Hence we can write equation (157) in the form

$$Curl^2\ \mathbf{F} = Grad\ (Div\ \mathbf{F}) - \nabla^2\mathbf{F}. \qquad . \quad (159)$$

Therefore in the case of any vector of which the divergence is zero we have

$$Curl^2\ \mathbf{F} = -\nabla^2\mathbf{F} = -Lap\ \mathbf{F}. \qquad . \quad (160)$$

This last is a very important equation.

In any region in which there are masses or electrified bodies which exert a force which varies inversely as the square of the distance, we can establish a relation between the density of the matter at any point and the Laplacean function of the potential as follows:

Consider any small particle of such attracting matter to be placed at the vertex of a cone of small angle, and let an oblique section of the cone be taken, the centre of that section being at a distance r from the vertex. Let A be the area of that oblique cross section of the cone and let a be the area of a normal section of the cone at that same distance from the vertex, that is a section perpendicular to the central axis of the cone.

Let θ be the angle between the normals to these two sections. Then $a = A \cos \theta$. If the mass of the attracting particle is m then the attracting force on a unit of matter placed at the centre of the normal section of the cone is m/r^2. This force is exerted perpendicularly to the normal section. Hence its component normal to the oblique section is $\frac{m}{r^2} \cos \theta$.

The product of this force and the area of the oblique section, called the surface integral of its normal force, is $A \frac{m}{r^2} \cos \theta$, and since $a = A \cos \theta$, this can be written $m \frac{a}{r^2}$.

Now the quotient of the area cut out from a spherical surface by a cone, with its vertex at the centre of the sphere, by the square of the radius is called the *solid angle* of the cone.

Hence for a cone of small solid angle $d\omega$ we have $d\omega = \frac{a}{r^2}$.

Since the surface of a sphere is $4\pi r^2$ the whole of the solid angles which can be arranged round a point is 4π.

If then a mass of attracting matter m is placed at any point, and if we describe round it any closed surface, and at each point of that surface resolve the force normally to the surface and take the product of this normal force F and an element of area of the surface, viz. dS, we can sum up or integrate this product for the whole surface or find the value of $\int F dS$. This surface integral of normal force must therefore be equal to 4π times the total quantity of attracting matter with the surface irrespective of its distribution. This is called Gauss' Theorem.

Let us apply this theorem to a small solid rectangle of sides, dx, dy, dz, parallel to the three axes.

Let the rectangle be filled with matter of density ρ. Then the amount of matter in the volume $dxdydz$ is $\rho dxdydz$.

Let the origin of co-ordinates be taken at the centre of this little rectangular volume and let the potential at that origin be V. Then the force in the direction of the x-axis at that point is $-\frac{dV}{dx}$.

Hence the force at the surface of the solid rectangle normal to x, viz. the surface $dydz$, at a distance $+\frac{1}{2}dx$ from the centre is $-\frac{dV}{dx}-\frac{1}{2}\frac{d^2V}{dx^2}dx$.

Also the force at the parallel surface but at an equal negative distance $-\frac{1}{2}dx$ is $-\left(-\frac{dV}{dx}+\frac{1}{2}\frac{d^2V}{dx^2}dx\right)$. The minus sign is prefixed to the bracket because the force is inwards at both surfaces as far as regards the attraction of the interior mass. Therefore the surface integral as far as these two surfaces are concerned is $-\frac{d^2V}{dx^2}dxdydz$. For the other two pairs of parallel surface it is $-\frac{d^2V}{dy^2}dydxdz$ and $-\frac{d^2V}{dz^2}dzdxdy$. Accordingly by Gauss' theorem we have the equation

$$4\pi\rho dxdydz=-\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}\right)dxdydz,$$

or

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)V+4\pi\rho=0, \quad . \quad . \quad (161)$$

or

$$Lap\ V+4\pi\rho=0. \quad . \quad . \quad (162)$$

This last equation is called Poisson's Equation and it enables us to calculate the density of matter or of electricity at any point in the field when we know the potential at that point. For

$$\rho=-\frac{1}{4\pi}\ Lap\ V. \quad . \quad . \quad (163)$$

The symbol ∇^2 is often used to denote $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}=Lap$, and Poisson's equation is thus written,

$$\rho=-\frac{1}{4\pi}\nabla^2V. \quad . \quad . \quad . \quad (164)$$

Accordingly at any place where there is no attractive matter or electricity the potential fulfils the condition

$$\nabla^2V=0. \quad . \quad . \quad . \quad (165)$$

This last equation, (165), is usually called Laplace's Equation.

In the next place, suppose that the distribution of the matter or electricity is given and that it is required to find the potential at every point in the field. We can find it as follows:

Consider one particle of matter with mass dm. Then at a distance r the force is $\frac{dm}{r^2}$. If a unit of matter is moved a distance dr against this force the work done is $\frac{dm}{r^2}dr$ or $dm\frac{dr}{r^2}$. Hence the whole work done in moving the unit up from infinity to that point is

$$dm\int_{\infty}^{r}\frac{dr}{r^2}=-\frac{dm}{r}.$$

This is called the potential dV at that point. Accordingly if we sum up this function for all the attractive matter in the field we have

$$V=\int\frac{dm}{r}. \quad . \quad . \quad . \quad (166)$$

If we denote dm by ρdv where ρ is the volume density and dv is an element of volume we have

$$V=\int\frac{\rho dv}{r}. \quad . \quad . \quad . \quad (167)$$

The process, therefore, of finding V from ρ as expressed by the above operation consists in summing up all the quotients of elements of mass ρdv by the distance r of that element of mass from the point at which it is desired to find the potential V, and this process may be called "potentializing" ρ or "potting" ρ. We can write it $V=Pot\ \rho$. On the other hand, the process of finding ρ when V is given, which is mathematically expressed by the operation

$$\rho=-\frac{1}{4\pi}\ Lap\ V,$$

may be called "lapping" V.

Hence we derive the following rules:

I. In an electric field or field in which matter exists which attracts inversely as the square of the distance, if we are given the volume density ρ of the matter at every point, we find the potential ∇ at any point by "potting" ρ.

II. If we are given the potential V at every point in the field we find the volume density of the matter at each point by "lapping" V at that point.

Hence the operations $\int\frac{\rho dv}{r}$ and $-\frac{1}{4\pi}$ *Lap V* are the inverse of each other, in the same sense in which $\frac{dy}{dx}$ is the inverse of $\int ydx$.

For non-divergent vectors we have shown that the operator *Curl*^{2} is equal to $-\nabla^2$ or to $-$*Lap*. Hence $-\frac{1}{4\pi}$ *Curl*^{2} is the inverse of *Pot* as an operator for such vectors.

17. Scalar and Vector Potentials

If there is any quantity which is a scalar quantity distributed in any way in a region or field, and if we divide up the space into small elements and divide the value of the scalar quantity in that small element of space by its distance from some fixed point, and sum up these quotients for the whole region, we obtain what is called the scalar potential at that fixed point. Thus, if the scalar quantity is *mass* and mdv is the element of mass in the elements of volume dv, then the sum of $\frac{mdv}{r}$ or $\int\frac{mdv}{r}$ is the scalar mass potential at the fixed point. If the scalar quantity is electric volume density ρ, then the integral $\int\frac{\rho dv}{r}$ is the electric scalar potential at the fixed point.

If, however, we are dealing with a distribution of a vector quantity, say, for instance, electric current density c, then the same integral $\int\frac{cdv}{r}$ gives us the vector potential. In this case the summation or integration is not effected by adding arithmetically the elements $\frac{cdv}{r}$, but by adding them vectorially.

If ρdv is a small quantity of electricity, and we bring it up to a point in a field at which the potential is V, then the work done is $V\rho dv$; and if V is due to another element of quantity $\rho' dv'$ at a distance r from the point, then $\frac{\rho dv\rho' dv'}{r}$ is a measure of the potential energy of the system. In the same way if ids and $i'ds'$ are two current elements at a distance r, then $\frac{i'ds'ids}{r}.cos\,\theta$, where θ is the angle between the current elements, is the potential energy of the system.

18. Line, Surface, and Volume Integrals

If we have any vector distributed in a field, and draw any line in that field, then at every point in that line the vector makes some angle θ with the line or tangent to the line; if a curve, let **F** be the vector at that point and F its size. Then $F \cos \theta$ is the component of the vector parallel to the curve or line. Let δs be an element of length or very short piece of that curve. Then the product $F \cos \theta \delta s$ is called a line element of the vector. If we sum up or integrate this product all along the curve or find the value of $\int F \cos \theta \delta s$, this is called the *line integral* of the vector taken along that line. If we take the line integral round a closed curve then we must always suppose ourselves to move in a counter clockwise direction (positive rotation) round the curve. For those elements of the line integral for which the resolved part of the vector is in the direction of the motion the product is to be regarded as positive in sign, but for those elements in which the resolved part of the vector is against the motion then that product is negative. Hence it may happen that the *line integral* is zero in going round a curve drawn in a vector field. When this is the case the *curl* of that vector is zero, and the vector itself is said to be *irrotational* and to be derived from a potential.

In the same manner, if we have a closed line bounding any surface drawn in a vector field we can suppose the surface divided into little elements of surface dS, and also the vector resolved perpendicularly to the surface at each point. Then the sum of all the products $F \cos \theta dS$ where θ is the angle between the vector and the normal to that surface or the integral $\int F \cos \theta dS$ is called the *surface integral of the vector.*

Again, we may have a scalar field within a closed surface, and we may divide up the space enclosed into little elements of volume dv, and we may sum up the product of the scalar at each point and the element of volume, and so obtain a *volume integral* of the type $\int S dv$.

There are two important theorems which are connected with the line integral of a vector, the surface integral of its curl and the volume integral of its divergence, which are as follows :—

Stoke's Theorem.—If there be a closed line drawn in any vector field the line integral of the vector taken round that curve is equal to the surface integral of the curl of that vector taken over any surface of which that line is the edge. For

the proof of this theorem we refer the reader to Coffin's book on *Vector Analysis.*

The Divergence Theorem.—If there be any closed surface drawn in a vector field which has divergence, then the surface integral of that vector taken over the surface is equal to the volume integral of the divergence of the vector taken throughout the enclosed space.

The truth of this theorem is obvious if we think of it in terms of lines of force. It merely states a truism, viz. that the algebraic sum of all the lines of force leaving a closed surface, counting those that go in negative and those that come out positive, must be equal to the sum total of all the sources of lines of force originating within that surface.

Since $\nabla \times \mathbf{F}$ is the curl of the vector $\mathbf{F}$, and $\nabla \cdot \mathbf{F}$ is the divergence of $\mathbf{F}$ we can state these two theorems symbolically as follows:—

(*Stoke's Theorem*) is $\int(\nabla \times \mathbf{F})dS = \int(\mathbf{F} \cdot \mathbf{t})ds, \quad . \quad . \quad (168)$

when $\mathbf{t}$ stands for unit vector along the tangent to a curve.

Also

(*The Divergence Theorem*) is $\int(\mathbf{F} \cdot \mathbf{n})dS = \int(Div\ \mathbf{F})dv,$
$= \int(\nabla \cdot \mathbf{F})dv, \quad (169)$

when $\mathbf{n}$ stands for unit vector along the normal to a surface, and dS is an element of surface, ds an element of a curve or line, and dv an element of volume.

Hence we can decide that a vector $\mathbf{B}$ is the curl of a vector $\mathbf{A}$ if we can prove that the surface integral of $\mathbf{B}$ over a surface is equal to the line integral of $\mathbf{A}$ round the boundary to that surface.

19. Hyperbolic Functions

We have shown that the ordinary circular trigonometrical functions *sin x*, *cos x*, etc. can be expressed in terms of exponentials with imaginary indices. Thus

$$sin\ x = \frac{\epsilon^{jx} - \epsilon^{-jx}}{2j}, \quad . \quad . \quad . \quad (170)$$

$$cos\ x \quad \frac{\epsilon^{jx} + \epsilon^{-jx}}{2}, \quad . \quad . \quad . \quad (171)$$

where $j = \sqrt{-1}$ and ϵ is the base of the hyperbolic logarithms.

If we remove from these formulæ the j we obtain expres-

sions for the so-called *hyperbolic functions* called respectively the hyperbolic sine and cosine, which are written *sinh x* and *cosh x*. Thus

$$sinh\ x = \frac{\epsilon^x - \epsilon^{-x}}{2}, \quad . \quad . \quad . \quad (172)$$

$$cosh\ x = \frac{\epsilon^x + \epsilon^{-x}}{2}. \quad . \quad . \quad . \quad (173)$$

Hence we have by division the hyperbolic tangent or

$$\frac{sinh\ x}{cosh\ x} = tanh\ x = \frac{\epsilon^x - \epsilon^{-x}}{\epsilon^x + \epsilon^{-x}}. \quad . \quad . \quad (174)$$

The reciprocal of *sinh x* is called the hyperbolic cosecant, and that of *cosh x* the hyperbolic secant, and that of *tanh x* is called the hyperbolic cotangent. Thus

$$\frac{1}{sinh\ x} = cech\ x, \quad . \quad . \quad . \quad (175)$$

$$\frac{1}{cosh\ x} = sech\ x, \quad . \quad . \quad . \quad (176)$$

$$\frac{1}{tanh\ x} = coth\ x. \quad . \quad . \quad . \quad (177)$$

Also $$sinh\ x + cosh\ x = \epsilon^x, \quad . \quad . \quad (178)$$

and $$cosh^2\ x - sinh^2\ x = 1, \quad . \quad . \quad (179)$$

where $sinh^2\ x$ stands for $(sinh\ x)^2$. . . (180)

If we expand the expressions (172), (173) by the exponential theorem we have

$$sinh\ x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \text{etc.}, \quad . \quad (181)$$

$$cosh\ x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \text{etc.} \quad . \quad . \quad (182)$$

Hence if we give to x any numerical value we can calculate the value of the functions *sinh x*, *cosh x*, etc.

In the above expressions

$$\left.\begin{array}{l} 2! \text{ stands for } 1 \times 2 = 2 \\ 3! \quad ,, \quad ,, \quad 1 \times 2 \times 3 = 6 \\ 4! \quad ,, \quad ,, \quad 1 \times 2 \times 3 \times 4 = 24 \\ 5! \quad ,, \quad ,, \quad 1 \times 2 \times 3 \times 4 \times 5 = 120 \end{array}\right\}, \quad (183)$$

etc. etc.

Hence $\sinh 1 = 1 + \frac{1}{6} + \frac{1}{120} + \text{etc.} = 1.17,$. (184)

and $\cosh 1 = 1 + \frac{1}{2} + \frac{1}{24} + \text{etc.} = 1.54.$. (185)

Tables of the values of some hyperbolic functions are given in Chapter XII.

CHAPTER II

UNITS, DIMENSIONS, AND SYSTEMS OF MEASUREMENTS

1. Fundamental Units

EVERY quantity which can be the subject of scientific investigation must be measured in terms of a unit or standard quantity of the same kind of magnitude, in such fashion that we can define the magnitude of the thing or property considered by stating that it is equal to so many multiples or such a fraction of the unit.

The *fundamental units* are the units of *length*, *mass*, and *time*.

Derived units are units for the measurement of other quantities which are based upon the fundamental units.

Absolute units are derived units referred directly to the fundamental units of length, mass, and time.

Arbitrary units are arbitrarily chosen units, having no special or simple relation to the fundamental units. Thus if we choose as our unit of force the weight of a mass of one pound, this is an arbitrary unit because it depends upon the locality on the earth, and upon the size and density of the earth.

If, however, we choose as the unit of force the force which after acting in a unit of mass for one unit of time gives it a speed of one unit of length per unit of time, this is an absolute derived unit of force.

The two chief systems of absolute measurement are (i.) that in which the unit of length is the centimetre (1 cm.), the unit of mass the gramme (1 grm.), and the unit of time the mean solar second (1 sec.); (ii.) that in which the unit of length is the British standard foot; the unit of mass the British standard pound, and the unit of time the mean solar second. The first is called the C.G.S. system and the second the F.P.S. system.

There are other systems in which the unit of length is the metre (1 m.), the unit of mass the kilogram (1 kgr.), and the unit of time the minute (=60 secs.).

The physical or actual standards of reference are the distances between certain marks on, or between the ends of bars of platinum iridium alloy for the standards of length; the masses of certain blocks of the same alloy as standards of mass; and a certain fraction of the period of one rotation of the earth on its axis as the standard of time.

Two standards of length and of mass are now legalised in Great Britain, viz. the *Imperial*, based on the British standard yard and British standard pound, and the *metric*, based on copies of the International Metre and International Kilogramme as preserved by the International Committee on Weights and Measures in a laboratory at Sèvres near Paris. The standard of time is derived from the Greenwich Observatory clock.

The relation between the Imperial and Metric Standards are as follows :—

1 metre	=1.09361426 British standard yards.
1 yard	=0.914399 metre.
1 kilogramme	=2.20462 British standard pounds.
1 pound	=0.453592 kilogramme.
1 sidereal day	=86164.09 mean solar seconds.
1 mean solar day	=86400 mean solar seconds by the Greenwich clock.

In the metric system all multiples and sub-multiples are decimal, and multiples are expressed by Greek prefixes and sub-multiples by Latin prefixes. Thus :—

mega	means	a million times.
kilo	,,	a thousand times.
hecto	,,	a hundred times.
deka	,,	ten times.
deci	,,	a tenth part of.
centi	,,	a hundredth part of.
milli	,,	a thousandth part of.
micro	,,	a millionth part of.

The following are therefore imperial and metric equivalents in units of length :—

1 metre	=1.093614	British standard	yards.
,,	=3.280842	,,	feet.
,,	=39.37011	,,	inches.
1 yard	=0.914399 metre.		
1 foot	=0.304797 metre.		
1 inch	=0.025400 metre.		

100 centimetres = 1 metre and hence

1 centimetre	= 0.393701 inch.
,,	= 0.032808 foot.
1 yard	= 91.4399 centimetres.
1 foot	= 30.4797 centimetres.
1 inch	= 2.5400 centimetres.
1 kilometre	= 1000 metres and hence
,,	= 1093.614 yards.
,,	= 3280.842 feet.
,,	= 0.6214 mile.
1 mile	= 160933 centimetres.
,,	= 1609.33 metres.
,,	= 1.60933 kilometres.

The following are approximate rules for rough calculations :—

2.5 centimetres or 25 millimetres	= 1 inch.
30.5 centimetres	= 1 foot.
3 feet 3 inches and $\frac{3}{8}$ of an inch	= 1 metre.
1 kilometre	= $\frac{5}{8}$ of a mile.
1 mile	= 5280 feet = 1.6 kilometres.
1 nautical or sea mile	= 6086 feet = (1$\frac{1}{8}$ standard mile nearly).

For superficial or Area measurement we have the following equivalents :—

1 square yard	= 8361.26 square centimetres.
1 square foot	= 929.03 sq. cms.
1 square inch	= 6.4515 sq. cms.
1 square metre	= 10.7639 square feet.

For Volume or Bulk measurements we have the following equivalents :—

1 cubic yard	= 764535 cubic centimetres.
1 cubic foot	= 28316 cubic cms.
1 cubic inch	= 16.387 cubic cms.
1 cubic metre	= 61024 cubic inches.
1 litre = 1 cubic decimetre	= 61.024 cubic inches.
	= 1.7598 pints.

The standard of capacity in British imperial measure is the *Imperial gallon*, which is the volume of 10 pounds avoirdupois of pure water at 62° Fahr., and is equal to 277.274 cubic inches.

Hence 1 cubic foot, which contains 1728 cubic inches, contains 6.2321 imperial gallons, and 1 cubic foot of water therefore weighs 62.321 imperial standard pounds at 62° Fahr.

The metric standard of capacity is the *litre*, which is 1 cubic decimetre nearly, and is exactly the volume of 1 kilogramme of pure water at 4° centigrade. The mass of 1 cubic decimetre of water at 4° C. is 0.9999707 kilogramme.

Hence we can obtain approximately the capacity in gallons of a rectangular cistern by multiplying the length, breadth, and depth in feet and this product by $6\frac{1}{4}$.

Also we can obtain the capacity in litres by multiplying the length, breadth, and depth in decimetres.

The following are the equivalents of the British imperial and metric units of mass :—

1 kilogramme	= 1000 grammes.
1 British Imperial pound	= 7000 grains.
1 kilogramme	= 2.20462 pounds.
1 gramme	= 15.432356 grains.
1 pound (avoir.)	= 0.453592 kilogramme.
1 pound (lb.)	= 453.59 grammes.
1 ton = 2240 lbs.	$= 1.01605 \times 10^6$ grammes.
1 metric ton	= 1 megagramme.
	= 1000 kilogrammes.

The ordinary British measures of liquid capacity are 1 gallon = 4 quarts = 8 pints. Hence 1 pint of water weighs $1\frac{1}{4}$ lbs. nearly. It contains 16 fluid ounces. One pound of water occupies a bulk of 0.8 pint nearly.

Examples.—

1. The diameter of the earth is roughly 8000 miles. What is its diameter roughly in kilometres and centimetres ?

Answer.—1 mile = 1.6 kilometres, hence the diameter is 12800 kilometres or 12.8×10^8 centimetres approximately.

Note.—The mean radius of the earth is actually about 6.37090×10^8 cms. A meridian quadrant of the earth is nearly 10 million metres or 10^9 centimetres.

2. Assuming the earth to be a sphere find its volume in cubic centimetres.

Answer.—The volume of a sphere is $\frac{4}{3}\pi r^3$, where r is the radius. Hence the volume of the earth in cubic centimetres is

$$\frac{4 \times 3.1416 \times (6.37)^3 \times 10^{24}}{3 \times (160933)^3} = 1.0832 \times 10^{27} \text{ cubic centimetres.}$$

3. If the mean density of the earth is 5.56 what is the mass of it in pounds ?

Answer.—The product of volume and density gives mass. Hence the mass of the earth is $1.08 \times 10^{27} \times 5.56$ grammes, and if this is divided by 1000 and multiplied by 2.204 it is reduced to pounds. Hence the mass of the earth is 13.224×10^{24} pounds.

2. Derived Units, Velocity, Acceleration, Force, Energy, and Power

The linear velocity of a particle at any instant is the rate at which its position is changing. If ds is an element of the path being described by the particle which is moved over in the time dt then the limiting value of $\frac{ds}{dt}$ is the velocity of the particle.

Hence if the velocity is uniform the velocity is measured by the space described in a unit of time.

Hence the unit of velocity is a rate of change of position of 1 centimetre per second or 1 foot per second. A velocity is therefore always expressed by some length divided by some time.

The *dimensions* of a velocity are therefore a *length* divided by a *time*. This is expressed in the form of a *dimensional equation* $V = LT^{-1}$. Suppose then that we are given a velocity reckoned in one set of units and we wish to find the numerical value of that same velocity reckoned in another set of units, we proceed as follows :—

Let v be the numerical value of the velocity when the unit of length is l and the unit of time t, then when the unit of length is l' and the unit of time t' the numerical value of that same velocity is $v \times \frac{\frac{l}{l'}}{\frac{t}{t'}}$. The reason is that the numerical value of the velocity is inversely as the size of the unit of length and directly as the size of the unit of time. Thus, for example, a train is moving with a velocity of 60 miles an hour. What is this expressed in feet per second?

We have in the first statement a unit of length of one mile $= 5280$ feet, and a unit of time of one hour $= 3600$ seconds, and in the second statement a unit of length of one foot and a unit of time of one second. Hence the answer is

$$60 \times \frac{\frac{5280}{1}}{\frac{3600}{1}} = 60 \times \frac{528}{360} = 88 \text{ feet per second.}$$

A velocity is therefore said to have dimensions LT^{-1} or $\frac{L}{T}$.

To find the multiplying factor to convert from measurement in one set of units to measurement in another set of units we

have to put instead of L the ratio of the units of length from which to which we convert, and for T the same ratio for the units of time.

For example.—What is the *change-ratio* to convert velocity from feet per second to centimetres per second?

Answer.—30.4, because 1 foot is 30.4 times 1 centimetre.

Acceleration is the rate at which velocity is changing at any instant. If v denotes velocity, then $\frac{dv}{dt}$ denotes acceleration or $\frac{d^2s}{dt^2}$ if ds denotes a very small element of length of the path described in a very small time dt. The dimensions of acceleration are therefore LT^{-2} or $\frac{L}{T^2}$, and are expressed in feet or centimetres per second per second.

Reckoned on the C.G.S. system, that is in centimetres and seconds, the acceleration of gravity is 981 centimetres per second per second. To find the number representing it when the foot is the unit of length we must put into the dimensional formula $\frac{L}{T^2}$ the ratio $\frac{1}{30.5}$, since 1 cm. is $\frac{1}{30.5}$th part of 1 foot, and hence the acceleration of gravity in feet per second per second is $\frac{981}{30.5} = 32.2$.

Momentum is the product of mass and velocity at any instant. Its dimensions are therefore $\frac{ML}{T} = MLT^{-1}$.

Force is measured by the rate at which the momentum of a body is changing at the instant considered. It is therefore expressed by $m\frac{dv}{dt}$ or by $m\frac{d^2s}{dt^2}$ in terms of increments of time, path, and velocity. The dimensions of force are $\frac{ML}{T^2}$ or MLT^{-2} as it is best written.

Force is therefore defined as the agency which changes the momentum of a body either in direction or in amount.

Newton's Laws of Motion may be paraphrased as follows:—

1. A body cannot change its own linear or angular momentum. If changed at all it must be by some external force or torque.

2. Force is measured by the rate of change of momentum in the direction of the impressed force.

3. If any force acts on two bodies the change in momentum in both of them produced by it is the same.

The unit of force on the C.G.S. system is called *1 Dyne*. It is the force which, after acting on a mass of 1 gramme for 1 second, gives it a velocity of 1 centimetre per second.

The unit of force on the F.P.S. system is *1 Poundal*. It is defined like the dyne, but with *pound* and *foot* substituted for *gramme* and *centimetre*.

Example.—How many dynes make 1 poundal?

Answer.—Since the dimensions of force are MLT^{-2}, and since 1 pound = 453.6 grammes and 1 foot = 30.5 cms. we have the change-ratio given by $\frac{453.6}{1} \times \frac{30.5}{1} = 13834$. Hence 1 poundal = 13834 dynes nearly, or more exactly 13825.

Example.—How many poundals are equal to the force called a "weight of 1 pound"?

Answer.—The weight of 1 pound mass is a force depending on the acceleration of gravity, and the force is equal to the product of mass and acceleration. Hence, since the acceleration of gravity is 32.2 feet per second per second the weight of 1 pound is equal to 32.2 poundals.

Note.—The poundal is an absolute unit of force on the F.P.S. system, but the "weight of 1 pound" is a gravitation and arbitrary or unsystematic unit, and varies with the locality.

It is necessary to distinguish between *linear* velocity, acceleration, momentum, and force, and *angular* velocity, acceleration, momentum, and torque or angular force.

In the expressions for the linear quantities we have m, which stands for the mass, and v, which stands for the linear velocity of the centre of mass or centre of gravity.

In the expressions for the angular equivalents we have to substitute for mass, the moment of inertia round the axis of rotation (K), and for an element of path ds, an element of angle $d\theta$.

The moment of inertia round any axis is defined as the sum of each element of mass of the body, each multiplied by the square of its perpendicular distance from the axis.

Hence the angular velocity is $\frac{d\theta}{dt}$, the angular acceleration $\frac{d^2\theta}{dt^2}$, the angular momentum $K\frac{d\theta}{dt}$, and the torque or angular force or impressed couple is $K\frac{d^2\theta}{dt^2}$.

Work or Energy.—If a body is displaced against the action of a force tending to move it in the opposite direction, *work* is said to be done on it or *energy* expended on it.

Thus if a mass m is raised a height h against the uniform acceleration of gravity g, work equal to mgh units is done on it.

The unit of work in mechanical engineering is the *foot-pound* and is the work done in lifting 1 lb. 1 foot against gravity. This, however, is dependent on the acceleration of gravity at the locality. The absolute unit of work in the F.P.S. system is the *foot-poundal* nearly equal to $\frac{1}{32 \cdot 2}$ part of 1 foot-pound.

The unit of work on the C.G.S. system is the *centimetre-dyne*, which is called 1 *erg*. Ten million *ergs* are called 1 *Joule*.

The dimensions of *work* are those of *force* multiplied by *length* or ML^2T^{-2}.

A common engineering unit of work is the *horse-power-hour* (H.P.H.). A horse-power (H.P.) is a rate of doing work equal to 33,000 foot-pounds per minute, or 550 foot-pounds per second. A horse-power-hour is the amount of work done when 1 horse-power continues for 1 hour or 3600 seconds. It is therefore equal to 1,980,000 foot-pounds.

It is therefore a gravitation unit and depends upon locality upon the earth. But 1 foot-pound is approximately equal to 32.2 foot-poundals, and hence 1 horse-power-hour is equal to $1{,}980{,}000 \times 32.2 = 63.75 \times 10^6$ foot-poundals.

Example.—How many *joules* are equal to 1 horse-power-hour?

Answer.—Since the dimensions of work are ML^2T^{-2}, and since 1 pound is 453.6 grammes, and 1 foot is 30.4797 centimetres, the change-ratio from F.P.S. to C.G.S. is $453.6 \times 30.4797 \times 30.4797 = 421390$. But 1 joule $= 10^7$ ergs, and 1 horse-power-hour $= 63.75 \times 10^6$ foot-poundals. Hence 1 horse-power-hour $= 6.375 \times 403961$ joules $= 2{,}575{,}281$ joules.

Power or *Activity* is the rate of doing work. It is measured by the work done per second, if uniform. The C.G.S. unit of power is therefore 1 erg done per second. A rate of working equal to 10^7 ergs per second is called 1 *watt*.

The F.P.S. unit of power is one foot-poundal per second.

At a place where the acceleration of gravity is 32.2 in F.P.S. measurement, 1 horse-power is 550×32.2 foot-poundals per second (nearly).

A preferable unit of power is the kilowatt $= 1000$ watts $= 1\frac{1}{3}$ horse-power (nearly).

The dimensions of *Power* are those of *Work* divided by *Time*, since it is the rate of doing work. Hence the dimensions of Power are ML^2T^{-3}. Hence the change-ratio from F.P.S. to C.G.S. system is the same as that for Work, viz. 421390.

Example.—How many watts make 1 horse-power?

Answer.—1 horse-power is 550 foot-pounds per second or 550×32.2 foot-poundals per second. 1 watt is 10^7 ergs per second. Hence 1 horse-power is $550 \times 32.2 \times 421390 \div 10^7 = 746$ watts.

TABLE OF EQUIVALENTS OF DERIVED UNITS IN THE C.G.S., F.P.S., AND GRAVITATION SYSTEMS OF MEASUREMENT.

Velocity.

1 foot per second = 30.4797 cms. per sec.
1 mile per hour = 44.704 cms. per sec.
1 kilometre per hour = 27.777 cms. per sec.
1 knot = 1 nautical or sea mile per hour = 6086 feet per hour. } = 51.453 cms. per sec.

Acceleration.

1 foot per second per second = 30.4797 cms. per sec. per sec.
Gravity = 32.2 feet per second per second = 981 cms. per sec. per sec.

Force.

1 pound weight = 32.2 poundals.
1 poundal = 13825 dynes.

Work.

1 foot-pound = 32.2 foot-poundals.
1 foot-poundal = 421390 ergs.
1 joule = 10^7 ergs.

Power.

1 horse-power = 550 foot-pounds per second.
1 watt = 10^7 ergs per second.
1 horse-power = 746 watts.

3. Electric and Magnetic Units

The systems of units for electric and magnetic quantities are based upon the C.G.S. system of absolute measurement starting from certain postulates.

Three such systems are in use: one based on a definition of the unit quantity of electricity, another based on a definition

of the unit magnetic pole, and a third on the practical system of measurement, making use of units derived from the magnetic system by the use of certain decimal multiples and submultiples.

The electric and magnetic systems are called absolute systems, because the units are based upon the fundamental units of length, mass, and time in the C.G.S. system of measurement.

The system of units based upon the definition of the unit of electric quantity is called the Electrostatic system of measurement, and the units the Electrostatic (E.S.) units.

In this system we have the following definitions :—

Quantity of Electricity.—The unit quantity of electricity is that quantity which if placed on a very small sphere will repel an equal quantity of the same sign placed upon a similar sphere with a force of 1 dyne, if the centres of the spheres are 1 centimetre apart. This is called the electrostatic unit of quantity.

This force depends upon the *dielectric constant* of the medium in which the spheres are placed, and will be less the greater the dielectric constant, provided the quantity remains the same. Hence in the above definition the spheres must be assumed to be in vacuo.

If Q is the quantity of electricity upon each of two equal spheres, and if their centres are at a distance d centimetres, and if the dielectric constant of the medium in which they are placed is K, then the expression for the force in dynes with which the spheres repel each other is $\frac{Q^2}{Kd^2}$. This being a force must be of the dimensions MLT^{-2}, and hence Q must be of the dimensions $M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}K^{\frac{1}{2}}$. The dimensions of K, the dielectric constant, are not known, so that we can only introduce it as a new and separate dimension.

Potential.—The potential at a point is measured by the work done in bringing up a unit of electricity to that point from infinity. Hence the unit of potential is such that to bring a unit of quantity placed on a small sphere up to that point will involve a unit of work. If V stands for potential, then the product QV is of the dimensions of work, or ML^2T^{-2}; but since the dimensions of Q are $M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}K^{\frac{1}{2}}$, the dimensions of V are $M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}$. This electrostatic unit of potential is equal to about 300 volts.

Current.—A current of electricity is measured by the quantity of electricity which flows across any section of the conductor per unit of time. Hence the unit electrostatic

current is a current which conveys one electrostatic unit of quantity per second across any section. This current is a very small one. It is only about 0.000333 of a microampere. The dimensions of current in the electrostatic system are those of a *quantity* divided by a *time* or are $M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}K^{\frac{1}{2}}$.

Capacity.—The capacity of a conductor is measured by the quantity of electricity required to charge it to unit potential, when it is infinitely far removed from all other conductors.

Hence if C denotes capacity, and Q and V quantity and potential, we have $Q = CV$.

Hence the dimensions of capacity in electrostatic measure are those of a quantity divided by those of a potential, expressed by

$$\frac{M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}K^{\frac{1}{2}}}{M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}} = LK.$$

Hence capacity is measured in terms of a length on the electrostatic system, and expressed as so many centimetres when the surrounding dielectric is air or vacuum.

Resistance.—The resistance of a conductor is numerically equal to the quotient of the potential difference between its ends by the current flowing through it when there are no internal sources of electromotive force. Hence the dimensions of resistance on the electrostatic system are those of a potential by a current, or

$$\frac{M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}}{M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}K^{\frac{1}{2}}} = L^{-1}TK^{-1}.$$

They are therefore of the dimensions of the reciprocal of a velocity.

Turning then to the magnetic system of units we find as the starting-point a definition of the unit magnetic pole as follows :—

Unit Magnetic Pole.—If two very long thin magnets have their opposite poles placed at a distance of 1 centimetre apart in air, and if they attract each other with a force of 1 dyne, they are said to be unit magnetic poles.

The force exerted by two magnetic poles depends on the permeability of the medium in which they are placed, and is inversely as this permeability. Hence if P is the strength of each pole, and d is their distance, and μ is the permeability of the medium in which they are placed, the force between the poles will be measured by $\frac{P^2}{\mu d^2}$.

Hence the dimensions of magnetic pole strength in this electromagnetic system of units is $M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}\mu^{\frac{1}{2}}$, for the same reasons as in the case of electric quantity measured on the electrostatic system.

Electric Current. — In the magnetic system of units a current is defined by the magnetic force exerted by it.

A unit current is defined to be one which, when flowing in a conductor having unit length, exerts a unit of force (1 dyne) upon a unit magnetic pole placed so as to be at a distance of 1 cm. from all parts of that circuit. The circuit must therefore be bent into a circular arc one radian in length, and the magnetic pole must be situated at the centre of that circle of unit radius.

This unit current is approximately equal to 10 amperes.

The force exerted by the current upon the pole is proportional to the current, to the length of the current, also to the pole strength, and inversely as the square of the distance of the pole from the current.

Hence if P stands for a pole strength, and I for a current strength, and S for the length of the conductor, and d for the distance of the pole from the current, the quantity $\frac{IPS}{d^2}$ must be of the dimensions of a force, viz. MLT^{-2}, but S and d are of the dimensions of a length, and P is of the dimensions $M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}\mu^{\frac{1}{2}}$ in the magnetic system. Therefore the dimensions of a current in this system must be

$$\frac{MLT^{-2}}{M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}\mu^{\frac{1}{2}}} = M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}.$$

Quantity of Electricity.—The unit of quantity on the electromagnetic system is the quantity of electricity conveyed by the unit of current in one unit of time.

Hence the dimensions of electric quantity in the magnetic system are

$$M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}.$$

This electromagnetic unit of electric quantity is equal to 10 coulombs or 10 ampere-seconds.

Potential.—The electromagnetic unit of potential is defined as the potential at a point such that the unit of work is done in bringing up to it one electromagnetic unit of quantity from an infinite distance.

The dimensions of potential are therefore those of a work $= ML^2T^{-2}$ divided by those of a quantity of electricity, viz.

$M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}$. Hence the dimensions of potential on the electromagnetic system are

$$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}\mu^{\frac{1}{2}}.$$

This unit of potential is such that 10^8 E.M. units are equal to 1 volt.

Resistance.—A conductor is said to have a unit of electrical resistance when it permits a current of one unit to flow through it when there is one unit difference of potential between its terminals, provided there are no internal sources of electromotive force. Hence the dimensions of resistance on the electromagnetic system are those of a potential divided by those of a current, viz.

$$\frac{M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}\mu^{\frac{1}{2}}}{M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}} = LT^{-1}\mu.$$

This unit of resistance is such that 10^9 E.M. units are equal to 1 ohm.

Capacity.—As in the case of electrostatic units the dimensions of capacity are those of a quantity divided by those of a potential, and hence in the electromagnetic system capacity has the dimensions, viz.

$$\frac{M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}}{M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}\mu^{\frac{1}{2}}} = L^{-1}T^{2}\mu^{-1}.$$

This unit of electric capacity is a very large one. It is equal to 10^{15} microfarads.

A consideration of the dimensions of these units on the two systems of measurement, electrostatic (E.S.) and electromagnetic (E.M.), shows that there is a definite relation between the sizes of the corresponding units in the two systems which is always a multiple or fraction of a certain unitary ratio. If we collect in a table the expressions for the dimensions of the various electric magnitudes we find them to be as follows :—

[TABLE

DIMENSIONAL EXPRESSIONS FOR VARIOUS ELECTRICAL QUANTITIES IN THE ELECTROSTATIC (E.S.) AND ELECTROMAGNETIC (E.M.) SYSTEMS OF UNITS.

Magnitude.	E.S.	E.M.	ratio $\frac{\text{E.S.}}{\text{E.M.}}$
Electric Quantity	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}K^{\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}$	$LT^{-1}\sqrt{\mu K}$
,, Current	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}K^{\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$	$LT^{-1}\sqrt{\mu K}$
,, Potential	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}\mu^{\frac{1}{2}}$	$L^{-1}T\frac{1}{\sqrt{\mu K}}$
,, Resistance	$L^{-1}TK^{-1}$	$LT^{-1}\mu$	$L^{-2}T^{2}\frac{1}{\mu K}$
,, Capacity	LK	$L^{-1}T^{2}\mu^{-1}$	$L^{2}T^{-2}\mu K$

In all cases the ratio is either of the dimensions of a velocity multiplied by $\sqrt{\mu K}$, or of the square of a velocity multiplied by μK, or of the reciprocal of a velocity multiplied by $\frac{1}{\sqrt{\mu K}}$, or of the reciprocal of the square of a velocity multiplied by $\frac{1}{\mu K}$.

But since the ratio of the sizes of units of the same quantity in different systems of measurement is a mere numeric, it follows that the quantity $\frac{1}{\sqrt{\mu K}}$ must be of the dimensions of a velocity. This is called the *unitary velocity* or ratio, and is denoted by c or by u. All these electrostatic and electromagnetic units as they stand are either too large or too small for practical use. Hence certain decimal multiples and submultiples have been adopted as the practical units.

4. Practical Electric Units

The unit of Current is called 1 *ampere*, and it is $\frac{1}{10}$th of the absolute electromagnetic unit of current.

The unit of Quantity is called 1 *coulomb*, and it is $\frac{1}{10}$th of the absolute E.M. unit.

The unit of Potenial is called 1 *volt*, and it is equal to 10^{8} absolute E.M. units.

The unit of Resistance is called 1 *ohm*, and it is equal to 10^{9} absolute E.M. units.

The unit of Capacity is called 1 *farad*, and it is equal to 10^{-9} of one absolute E.M. unit.

The farad is even too large for much use, and hence the microfarad or one-millionth part of a farad is the common unit of capacity.

A large number of investigations have been made to determine the value of the unitary ratio "u," when the centimetre and second are the units of length and time.

It can be shown theoretically that this velocity u should be the same as the velocity of light, no matter what units are used.

The velocity of light is 300,000 kilometres per second nearly. Hence in cms. and secs. $u = 3 \times 10^{10}$.

The value of u can best be experimentally determined by measuring a capacity both electrostatically and electromagnetically.

Thus, to take a simple instance, suppose a sphere of radius r cms. were charged with electricity when in vacuo. We have shown that the dimensions of a capacity are a *length*. Each part of the charge of a sphere is at the same distance from the centre. Hence if Q is the whole charge of the sphere, $\frac{Q}{r}$ is the potential of that charge at the centre.

If the sphere is a conductor all parts of the sphere must be at the same potential. Hence the potential is $\frac{Q}{r}$, but the potential of the sphere is also expressed by $\frac{Q}{C}$ where C is the capacity. Hence $C = r$, or the capacity is expressed in electrostatic units by a number equal to the radius of the sphere in centimetres.

Suppose then that we charge the sphere to a potential V volts by means of a battery and discharge it to earth N times a second through a wire of resistance R ohms.

Let C' be the capacity of the sphere reckoned in *farads*, then the quantity imparted to the sphere is $C'V$ coulombs at each charge, and the average current in the discharge wire is $NC'V$ amperes. But the current in the wire is by Ohm's law equal to $\frac{V}{R}$.

Hence we have the equation

$$NC'V = \frac{V}{R},$$

or

$$C' = \frac{1}{NR}.$$

But now the capacity in electrostatic measure C is represented by r the radius of the sphere. Hence the ratio

$$\frac{C}{C'} = NRr.$$

But we have seen that the ratio of the numerical value of a given capacity when measured in electrostatic units to its value in electromagnetic units is given by $u^2 = \frac{1}{\mu K}$. Hence we have

$$\frac{C}{C'} = NRr = u^2,$$

or

$$u = \sqrt{NRr}.$$

Accordingly, if we can measure the radius of the sphere in centimetres and the number of times it is discharged per second and the resistance in electromagnetic units of the wire through which it is discharged, we can find u.

The resistance of the wire can be measured in ohms, and 1 ohm is 10^9 absolute electromagnetic units of resistance.

The experiment therefore consists in making a comparison between the current produced through a wire of resistance R ohms by means of a given steady voltage, and the current produced when the sphere is charged by the same voltage and discharged N times per second through the same wire.

Some of the measurements which have in this manner been made of the value of u in centimetres and seconds are given in the following table:—

TABLE OF OBSERVED VALUES OF THE UNITARY RATIO u OBTAINED BY DIFFERENT EXPERIMENTALISTS.

Name of Experimentalist.	Value of Unitary Ratio.
Himstedt	3.0057×10^{10} cms. per sec.
Rosa	3.0000×10^{10} ,, ,,
Thomson and Searle	2.9960×10^{10} ,, ,,
H. Abrabam	2.9913×10^{10} ,, ,,
Pellat	3.0092×10^{10} ,, ,,
Hurmuzescu	3.0010×10^{10} ,, ,,
Perot and Fabry	2.9978×10^{10} ,, ,,
Mean value	3.0001×10^{10} ,, ,,

The value of u is therefore taken as 3×10^{10} cms. per second.

The mean of the best results for the velocity of light in

vacuo is 2.99852×10^{10} centimetres per second or 299,852 kilometres per second, and this is probably correct to within 1 part in 10,000.

Hence we have the following values for the electrostatic units and for the electromagnetic units in terms of the practical units :—

1 electrostatic unit of quantity $= \frac{1}{3 \times 10^9}$ of a coulomb.
1 electromagnetic unit of quantity $= 10$ coulombs.
1 electrostatic unit of current $= \frac{1}{3 \times 10^9}$ of an ampere.
1 electromagnetic unit of current $= 10$ amperes.
1 electrostatic unit of potential $= 300$ volts.
1 electromagnetic unit of potential $= \frac{1}{10^8}$ of a volt.
1 electrostatic unit of resistance $= 9 \times 10^5$ megohms.
1 electromagnetic unit of resistance $= \frac{1}{10^9}$ of an ohm.
1 electrostatic unit of capacity $= \frac{1}{9 \times 10^5}$ of a microfarad.
1 electromagnetic unit of capacity $= 10^{15}$ microfarads.

The ratio of the sizes of the electrostatic to electromagnetic unit in the case of corresponding quantities are u^2 for resistance, u for potential, $\frac{1}{u}$ for quantity and current, and $\frac{1}{u^2}$ for capacity.

5. The Definition of the Practical Units for Electrical Measurement

The following definitions of the practical electrical units have been the subject of international agreements and were finally settled at an Electrical Congress held in London in October 1908 :—

1. The practical unit of electric resistance is the *International Ohm*. It is the resistance offered to an unvarying current of electricity by a column of pure mercury at the temperature of melting ice having a mass of 14.4521 grammes, a constant cross section and a length of 106.3 centimetres.

2. The practical unit of electric current is the *International Ampere*. It is the unvarying current which, when passed through a neutral solution of nitrate of silver, deposits silver at the rate of 0.00111800 of a gramme per second.

3. The practical unit of electromotive force or potential difference is the *International Volt.* It is that electromotive which, when steadily applied to a conductor having a resistance of 1 international ohm, creates in it a current of 1 international ampere. It is for practical purposes represented by $\frac{10000}{10184}$ of the electromotive force of a Cadmium Weston cell at 20° C. prepared in the manner described in a certain specification.

4. The practical unit of electric quantity is the *International Coulomb.* It is the quantity of electricity transferred in one second by a current equal to the international ampere.

5. The practical unit of capacity is the *International Farad.* It is the capacity of a conductor which is charged to a potential of one international volt by imparting to it a quantity of one international coulomb.

6. The practical unit of inductance is the *Henry.* It is the inductance of a circuit in which an induced electromotive force of 1 international volt is created when the current in it varies at the rate of 1 international ampere per second.

7. The practical unit of work or energy is the *Joule*, which is represented by the work suspended per second on a circuit in which there is a current of 1 international ampere, and between the ends of which there is a difference of potential of 1 international volt.

8. The practical unit of power is the *Watt.* It is a rate of working equal to 1 joule per second.

The practical method of measuring a magnetic field is in *lines* or *kilolines.* The *line* is identical with 1 electromagnetic unit of magnetic flux, and the *kiloline* is 1000 lines.

For practical and portable purposes the ohm and its multiples or submultiples are represented by the resistance of coils of wire generally constructed of manganin, platinum-silver, or constantan, or some other alloy having a small variation of resistance with temperature.

In the case of a solid material we may consider either the resistance of a unit cube, viz. a cube having a side of 1 centimetre to conduction between opposed forces, or we may consider the resistance of a wire of a certain length, say 1 metre, and weighing 1 gramme.

The former is called the *volume resistivity* or *specific resistance*, and the latter the *mass-resistivity.*

The *resistivity* (ρ) of materials is generally stated in ohms

per centimetre cube, or it may be in absolute electromagnetic units per centimetre cube at 0° centigrade.

In the case of a wire of uniform section s and length l and volume resistivity ρ at 0° C., the resistance of the wire at 0° C. is

$$R_0 = \frac{\rho l}{s}.$$

The resistance at $t°$ denoted by R_t is given by the formula

$$R_t = R_0(1 + \alpha t)$$

where α is the temperature coefficient.

The following is a table of the volume resistivities of pure metals from the experimental results of Professors J. A. Fleming and Sir James Dewar (see *Phil. Mag.*, September 1893):—

Metal.	Volume Resistivity (ρ) in Electro-magnetic Units (C.G.S.) per Centi-metre Cube.	Mean Tempera-ture Coefficient (α) between 0° C. and 100° C.
Silver (pure and well annealed)	1468	0.00400
Copper (pure and well annealed)	1561	0.00428
Gold (annealed)	2197	0.00377
Aluminium (annealed)	2665	0.00435
Magnesium (pressed)	4355	0.00381
Zinc	5751	0.00406
Nickel (electrolytic)	6935	0.00618
Iron (annealed)	9065	0.00625
Cadmium	10023	0.00419
Palladium	10219	0.00354
Platinum (annealed)	10917	0.00367
Tin (pressed)	13048	0.00440
Thallium (pressed)	17633	0.00398
Lead (pressed)	20380	0.00411
Bismuth (electrolytic)	110000	0.00433
Mercury (pure)	94070	0.00098

With regard to the above table of resistivities it should be noted that the results here given are for very pure metals in a well-annealed or soft condition. The metals in their ordinary commercial purity and not well annealed have considerably higher resistivities.

A very small amount of impurity will very greatly increase the resistivity.

The resistivities can also be stated in ohms per metre of

a wire of uniform cross section of such size as to weigh 1 gramme.

Thus the resistivity of pure annealed copper is defined in terms of a standard called Matthiessen's Standard, as follows:—

Matthiessen's Standard for annealed high conductivity commercial copper shall be considered to be a wire of pure annealed copper one metre long and weighing one gramme whose resistance at 60° *F. is* 0.150822 *international ohms.*

If the copper is hard-drawn, which is defined by stating that a hard-drawn wire is one which will not elongate more than 1 per cent without fracture, then the definition is as follows:—

Matthiessen's Standard for hard-drawn high conductivity commercial copper shall be considered to be a wire of pure hard-drawn copper one metre long and weighing one gramme whose resistance at 60° *F. is* 0.153858 *international ohms.*

The conversion from resistance at 60° F. to resistance at 32° F = 0° C. is effected by the use of Matthiessen's temperature coefficient for copper as follows:—

$$R_t = \frac{R_{32}}{1 - 0.00215006\,(t-32) + 0.00000278\,(t-32)^2}$$

The *mass-resistivity* of copper, that is the resistance of a wire a metre long and weighing one gramme at 0° C., is therefore

For hard-drawn copper . .	0.1449 ohm.
For annealed copper . .	0.1421 ohm.

To convert these values to volume resistivities in C.G.S. units per centimetre cube let ρ be the volume resistivity and ρ' the metre-gramme or mass-resistivity. Let d be the density of copper, which is taken to be 8.912 grammes per centimetre cube, 550 pounds per foot cube.

Then 1 gramme of copper occupies a volume of $\frac{1}{8.912}$ centimetre cube and for a wire 100 centimetres long and weighing 1 gramme, the section s is given by the equation

$$100\,s = \frac{1}{8.912}$$

or

$$s = \frac{1}{100 \times 8.912}.$$

Accordingly, if ρ is the volume resistivity in electromagnetic units per centimetre cube, and if ρ' is the resistance per metre-gramme reckoned in ohms, we have the equation

$$10^5\rho' = \rho 10^4 \,.\, 8.912,$$

or

$$\rho = \rho' \frac{10^5}{8.912}.$$

Now for annealed copper at 0° C. we have $\rho' = 0.1421$ ohm and for hard-drawn copper $\rho' = 0.1449$ ohm. Hence the volume resistivities are found by multiplying these numbers by 100000 and dividing by 8.912. This gives the following rules :—

The volume resistivity of pure hard-drawn copper at 0° *C. is* 1.626 *microhms per centimetre cube or* 1626 *C.G.S. units. The volume resistivity of pure annealed copper at* 0° *C. is* 1.594 *microhms per centimetre cube or* 1594 *C.G.S. units.*

The calculation of the resistivity at any other temperature requires the use of the temperature coefficient. It has been decided that the average temperature coefficient of pure copper between 30° F. and 100° F. or between 0° C. and 40° C. shall be given by the following formula,

$$\rho_t \text{° C.} = \rho_0 \text{° C.}(1 + 0.00428\, t),$$

where t is the centigrade temperature and ρt° C. and ρ0° C. signify the resistivities at t° and 0° centigrade.

Provided with these constants the resistivity of a copper wire of any size and at any temperature can be determined.

Example.—Find the resistance of an annealed copper wire 1000 yards long, No. 10 standard wire gauge (S.W.G.) in size, at 60° F.

Answer.—The diameter of a No. 10 S.W.G. wire is 0.128 inch or 3.25 millimetres.

The diameter of the wire is therefore 0.325 cm. The cross-sectional area of the wire is then $\frac{\pi}{4} \times (0.325)^2 = 0.083$ square centimetre.

The length of the wire is 1000 yards and 1 yard = 91.4392 cms.

Hence 1000 yards = 91439.2 cms.

The resistivity of annealed copper is 1594 C.G.S. units at 0° C. Hence the resistance of the wire at 0° C. is

$$\frac{1594 \times 91439}{10^9 \times 0.083} = 1.756 \text{ ohms.}$$

Hence its resistance at 60° F. = 15° C. is

$$1.756\,(1 + 0.00425 \times 15) = 1.868 \text{ ohms.}$$

In this manner Tables have been calculated for the resistances of certain lengths of standard sizes of copper wire, and some of these data are given in the tables in Chapter XII. of this pocket-book.

CHAPTER III

HIGH FREQUENCY RESISTANCE AND INDUCTANCE MEASUREMENT

1. Difference between High Frequency Resistance and Steady Resistance

If a uniform or steady unidirectional current of A amperes flows through a conductor of resistance R ohms the power dissipated in the conductor is RA^2 watts. If this is denoted by W we may define the resistance of the conductor as the ratio of W to A^2 or numerically as equal to $\frac{W}{A^2}$.

It is easy to show that this ratio is a minimum for uniform distribution of the current over the cross section of the conductor. For if we consider the uniformly distributed current as divided into filamentary currents, each flowing in an elementary strand or filament into which we may consider the conductor divided, then it is evident that if an element of current is transferred from one filament to another it will make no difference in the total current, and therefore no difference in the total current squared. On the other hand, taking the current away from one filament, and putting it into another would increase the total power expended in the conductor, because it would make the heat generated zero in that filament in which the current was zero and four times its original value in that filament in which the current was doubled. Hence the ratio $\frac{W}{A^2}$ would be increased by any redistribution of the current over the cross section of the conductor non-uniform or less symmetrical than before. Hence the resistance of a conductor is determined by the mode of distribution of the current over the cross section of the conductor.

It can be shown by theory and experiment that a high

frequency alternating current is confined to a thin outer layer or skin in the case of a good conductor, such as a copper wire.

2. Penetration of an Alternating Current in a Conductor

We can calculate the thickness of this current skin in certain simple cases as follows :—

Let us suppose a slab of the conductor has a flat surface, and is infinitely extended downwards, lengthwards, and in breadth. Consider a very small cube of the material at some depth x below the surface. Imagine an alternating electromotive force to act on the material parallel to the flat surface and create an alternating current in it. There will therefore be a greater current density at the surface than at various depths below it. Let us consider the state of affairs in that small cube of the material at any instant. There is a current in it flowing say in a direction y and flowing perpendicular to the face xz. This current creates a magnetic force, and by a fundamental property the line integral of that force is equal numerically to 4π times the enclosed current.

Let C be the current density in the small cube. Let δx, δy, δz be the sides of that cube, and let the current flow parallel to δy and perpendicular to δx, δz. Then, since C is the current per square centimetre or current density, the total current flowing through the cube is $C\delta x\delta z$. Again there is a magnetic force H parallel to the side δz at the level x below the surface. Hence the magnetic force at a depth $x+\delta x$ is $H+\frac{dH}{dx}\delta x$ (see Chap. I., end of § 11).

There is no magnetic force parallel to the edge δx. Hence the line integral of the magnetic force round the edge of the face $\delta x\delta z$ is $-\frac{dH}{dx}\delta x\delta z$, and this must be equal to $4\pi C\delta x\delta z$, or

$$4\pi C = -\frac{dH}{dx} \quad . \quad . \quad . \quad (1)$$

Then again, since the current is an alternating current, there will be a variation in the electromotive force caused by the time rate of decrease of the total magnetic flux through the face $\delta x\delta y$ of the cube.

This rate is expressed by $-\mu\frac{dH}{dt}\delta x\delta y$, where H is the magnetic force, and μ is the magnetic permeability of the

material. Hence by Faraday's law this must be equal to the electromotive force induced round the face $\delta x\delta y$ of the cube. This last is equal to the line integral of the electric force or to the difference between the electric force E at a depth x and its value $E+\frac{dE}{dx}\delta x$ at a depth $x+\delta x$ multiplied by the length δy of the side of the cube.

Hence
$$-\mu\frac{dH}{dt}\delta x\delta y=\frac{dE}{dx}\delta x\delta y,$$

or
$$-\mu\frac{dH}{dt}=\frac{dE}{dx}. \quad . \quad . \quad . \quad (2)$$

Now by Ohm's law we have $C=\frac{E}{\rho}$, when ρ is the volume resistivity, because C is the current density or current per square centimetre.

Substituting, therefore, $C\rho$ for E in (2) we have
$$-\frac{dH}{dt}=\frac{\rho}{\mu}\frac{dC}{dx}, \quad . \quad . \quad . \quad (3)$$

and from (1)
$$-\frac{dH}{dx}=4\pi C. \quad . \quad . \quad . \quad (4)$$

Differentiating (3) with respect to x and (4) with respect to t and subtracting, we have
$$\frac{d^2C}{dx^2}=\frac{4\pi\mu}{\rho}\frac{dC}{dt}. \quad . \quad . \quad . \quad (5)$$

Before we can deal with the solution of the above equation we must consider how a quantity which varies as a sine function of the time can be differentiated with regard to time as the variable.

If the current is an alternating current we may assume that C varies as a sine function of the time. Thus
$$C=C_0 \sin pt, \quad . \quad . \quad (6)$$

where C_0 is the maximum value of the current density, and $p=2\pi n$, where n is the frequency and C is the current density at any time t.

If we differentiate (6) with regard to time we have
$$\frac{dC}{dt}=pC_0 \cos pt,$$

and if we differentiate again we have

$$\frac{d^2C}{dt^2} = -p^2C_0 \sin pt = -p^2C. \qquad (7)$$

Hence the second differential of C is obtained by multiplying C by $-p^2$ or by $(jp)^2$, where $j = \sqrt{-1}$. Hence jp must be equivalent as an operator to $\frac{d}{dt}$ because when twice repeated they both produce the same result when operating on a sine function of the time, since then

$$\frac{d}{dt}\cdot\frac{d}{dt} = \frac{d^2}{dt^2}, \text{ and } (jp)\cdot(jp) = -p^2.$$

Accordingly, we have the general rule that when dealing with simple periodic time functions the operation $\frac{d}{dt}$ can be replaced by multiplication by jp, or by $2\pi nj$, or by $2\pi j\frac{1}{T}$, where n is the frequency and T the period. Hence in the present case we can write the equation (5) in the form

$$\frac{d^2C}{dx^2} = jp\frac{4\pi\mu}{\rho}C = \left\{(2\pi)^2\frac{\mu n}{\rho}2j\right\}C. \quad . \quad (8)$$

But $2j = 2\sqrt{-1} = (\sqrt{-1}+1)^2 = (j+1)^2$.

Hence (8) becomes

$$\frac{d^2C}{dx^2} = \left\{(2\pi)^2(j+1)^2\frac{\mu n}{\rho}\right\}C, \quad . \quad . \quad (9)$$

which may be written in the form

$$\frac{d^2C}{dx^2} = (\alpha+j\alpha)^2C, \quad . \quad . \quad . \quad (10)$$

where

$$\alpha = 2\pi\sqrt{\frac{n\mu}{\rho}} \text{ or } 2\pi\sqrt{\mu n\sigma}. \quad . \quad . \quad (11)$$

The equation (10) is a differential equation giving us the current density at any depth x. We have already seen (see Chap. I. § 15, equation (113)) that a solution of the differential equation

$$\frac{d^2y}{dx^2} = a^2y$$

is

$$y = Ae^{-ax} + Be^{+ax}. \quad . \quad . \quad (12)$$

In the case considered the current will clearly be zero at an infinite depth, therefore the constant $B=0$, and the solution of (10) is

$$C = C_0 \epsilon^{-\alpha x}(\cos \alpha x - j \sin \alpha x), \quad . \quad . \quad (13)$$

where C_0 now stands for the maximum value of the current density at the surface of the slab.

Hence we see that the current density diminishes in a geometrical ratio as the depth x increases in arithmetical ratio, and the attenuation of the current with depth is determined by the factor $\epsilon^{-\alpha x}$, where $\alpha = 2\pi\sqrt{\frac{n\mu}{\rho}}$.

Since ϵ^{-4} is a small fraction equal to about 0.02, it follows that when the depth x is such that

$$x = \frac{2}{\pi}\sqrt{\frac{\rho}{n\mu}} = \frac{0.636}{\sqrt{n\mu\sigma}}, \quad . \quad . \quad (14)$$

where σ is the conductivity, then the current will be only 2 per cent of the value at the surface. Hence the depth x is the practical thickness of the current *skin* in the conductor. This can be calculated when we are given the numerical values of ρ, n, and μ.

Example.—To what depth does an alternating current having a frequency of one million penetrate into a copper slab?

Answer.—The magnetic permeability (μ) of copper is unity, and its specific resistivity ρ is 1600 C.G.S. units. Hence

$$\rho = 1600, \ \mu = 1, \ n = 10^6.$$

Therefore by equation (14) the thickness of the current skin is

$$x = \frac{0.636\sqrt{1600}}{\sqrt{10^6}} = .02544 \text{ centimetre},$$

or x is $\frac{1}{4}$ millimetre nearly.

Example.—To what depth does an alternating current having a frequency of one million penetrate into an iron slab?

Answer.—In this case $n = 10^6$ and $\rho = 10000$ (nearly) C.G.S. units. The permeability μ for iron under a frequency of 10^6 is somewhat ill-defined, but it is certainly as high as

1000. Suppose we take $\mu = 1000$. Then the thickness of the current skin for iron is

$$x = \frac{0.636 \times \sqrt{10^4}}{\sqrt{10^6 . 10^3}} = 0.002 \text{ centimetre (nearly).}$$

Hence x is about $\frac{1}{50}$th of a millimetre.

Accordingly, the current penetrates to a depth in copper greater by about 13 times than in the case of iron.

3. Calculation of the High Frequency Resistance of Wires

Hence in consequence of the fact that a high frequency alternating current is confined to a thin skin on the surface of a copper or metal wire the electrical resistance of the metal to that current is very different from, and much greater than, its resistance to steady currents as proved in § 1 of this chapter.

Formulæ have been given as follows for the high frequency resistance of round-sectioned straight wires. Since the wires used are generally of non-magnetic material we take $\mu = 1$ in all formula.

For a round wire the *discriminating constant* is denoted by h such that

$$h = \frac{n}{\rho} c^2, \quad . \quad . \quad . \quad (15)$$

where n = frequency, ρ = volume resistivity, and $c = \pi d$ is the circumference of the wire.

In place of h a constant q is often used, such that

$$q = \sqrt{2h} = \pi d \sqrt{\frac{2n}{\rho}}. \quad . \quad . \quad (16)$$

If h is less than unity for the wire and frequency in question then the high frequency resistance R' is calculated from the steady or ohmic resistance R by a formula given by Lord Rayleigh, viz. :—

$$R' = R\left(1 + \frac{h^2}{48} - \frac{h^4}{2880} + \frac{h^6}{58647} \ldots\right). \quad (17)$$

If, however, h is greater than 5 then the above formula is unsuitable, and we must use a formula given by Dr. A. Russell, which is

$$R' = R\left(\frac{\sqrt{h}}{2} + \frac{1}{4} + \frac{3}{32\sqrt{h}} - \frac{1}{16h\sqrt{h}}\right), \quad . \quad (18)$$

which can generally be abbreviated to

$$R' = R\left(\frac{\sqrt{h}}{2} + \frac{1}{4}\right). \quad . \quad . \quad (19)$$

In the above formula R stands for the steady resistance of a circular-sectioned wire of resistivity ρ and diameter d and length l.

Hence $$R = \frac{4\rho l}{\pi d^2}. \quad . \quad . \quad . \quad (20)$$

Now $$h = \frac{n}{\rho}\pi^2 d^2 \text{ and } \sqrt{h} = \pi d\sqrt{\frac{n}{\rho}}.$$

Hence the high frequency resistance R' of a straight or nearly straight wire of length l, diameter d, and resistivity ρ, is given by

$$R' = \frac{4\rho l}{\pi d^2}\left(\frac{\pi d}{2}\sqrt{\frac{n}{\rho}} + \frac{1}{4}\right) \quad . \quad . \quad (21)$$

$$= \frac{2l}{d}\sqrt{n\rho} + \frac{\rho l}{\pi d^2}. \quad . \quad . \quad (22)$$

This is of course in C.G.S. electromagnetic units, if l and d are measured in centimetres and ρ in E.M. units. It must be divided by 10^9 to reduce to ohms.

Example.—For a frequency of one million and for a copper wire No. 10 S.W.G. size, how much greater is the high frequency resistance than the ordinary or steady resistance?

Answer.—The diameter of No. 10 S.W.G. wire is 0.128 inch or 0.325 cm. (see Chap. XII., Wire Tables).

Hence we have $n = 10^6$, $d = 0.325$, and for copper $\rho = 1600$.

The circumference of the wire $c = \pi d$ is then $3.1416 \times 0.325 = 1.02$ cm.

Therefore $\sqrt{h} = 1.02\sqrt{\frac{10^6}{1600}} = 25$ nearly. Accordingly by formula (19) we have

$$\frac{R'}{R} = \frac{25}{2} + \frac{1}{4} = 12.75,$$

and the resistance of this wire for this frequency is 12.75 times the steady resistance. The expression (19) enables us to answer the following question: What is the maximum size of wires of given resistivity ρ which has sensibly the same resistance for high frequency currents of a certain frequency as

for steady currents? We have then in the equation (19) to put $R' = R$ and we have

$$1 = \frac{\sqrt{h}}{2} + \frac{1}{4}. \quad . \quad . \quad . \quad (23)$$

Therefore $\sqrt{h} = 1.5$, but $\sqrt{h} = \sqrt{\frac{n}{\rho}}\pi d$, and hence the required diameter of the wire is given by the expression

$$d = \frac{1.5}{\pi}\sqrt{\frac{\rho}{n}} = 0.4774\sqrt{\frac{\rho}{n}}. \quad . \quad (24)$$

Example.—What is the largest size of copper wire which can be assumed to have the same resistance to alternating currents of a frequency of one million as to steady currents?

Answer.—In this case $\rho = 1600$ and $n = 10^6$.

Hence $$d = 0.4774\sqrt{\frac{1600}{10^6}} = 0.0191 \text{ cm}$$

The wire must therefore not exceed 0.19 mm. in diameter, or about $\frac{1}{5}$th mm. or $\frac{1}{125}$ of an inch. The nearest S.W.G. size is No. 36.

The higher the resistivity the larger the maximum size of wire.

Dr. L. W. Austin has calculated a table showing for wires of various sizes the maximum size which can be used for certain frequencies.

The above formulæ are only applicable to circular-sectioned and nearly straight wires made of non-magnetic material and for such frequencies and resistivities that h is greater than about 5.

If the wire is in the form of a spiral, then the reaction of the different turns on each other causes the current skin to be unequally thick at all points of the circumference of the wire.

The current is most concentrated on the inner side of the spiral. Hence in consequence of the principle explained in § 1 of this chapter there is a still further increase to alternating currents. Not only has a wire a greater resistance to alternating currents than to steady currents, but a spiralised wire has a greater resistance to alternating currents than the same wire when stretched out straight.

The formulæ which have been derived to express this spiral resistance are not very consistent or simple. Almost the only case which can be dealt with analytically is that of a

spiral of bare wire of one layer; the different turns of wire are almost in contact and the wire of square section.

Let R'' be the resistance of such a spiral, and let R' be the resistance of the wire when stretched out straight, and let R be its resistance to steady or unidirectional currents. For this case a formula has been given by L. Cohen as follows (see J. A. Fleming, *Principles of Electric Wave Telegraphy and Telephony*, 2nd ed. chap. ii. § 1, and also *Bulletin of the Bureau of Standards*, Washington, U.S.A., vol. iv. No. 1):

$$\frac{R''}{R} = 1 + 4.8N^2d^3\sqrt{\frac{n}{\rho}}, \quad . \quad . \quad (25)$$

where d is the diameter of the wire, ρ the resistivity, N the number of turns of the spiral per centimetre of length, and n is the frequency.

For example.—Suppose a spiral made of copper wire wound with four turns per cm. of length of the spiral, and let the diameter of the wire be 2 mm. and the frequency of the oscillations one million, what is the ratio of high frequency resistance to steady resistance?

Answer.—We have $d = 0.2$ cm., $n = 10^6$, $\rho = 1600$, $N = 4$, and hence by (25)

$$\frac{R''}{R} = 1 + 4.8 \times (4)^2 \times (0.2)^3\sqrt{\frac{10^6}{1600}} = 1 + 15.36 = 16.36.$$

If the wire were stretched out straight its alternating current resistance R' would be given by formula (19), viz.:

$$\frac{R'}{R} = \frac{\sqrt{h}}{2} + \frac{1}{4} = 8.1.$$

Therefore

$$\frac{R''}{R'} = \frac{16.36}{8.1} = 2.02 \text{ nearly.}$$

Hence this wire when spiralised has about twice the resistance to currents of a frequency of one million that it would have if stretched out straight. It must be noticed that Cohen's formula (25) has been deduced on the assumption that the product Nd is nearly unity, and therefore only applies to spirals of one layer and of closely adjacent turns.

4. Measurement of the High Frequency Resistance of Wires

In consequence of the limited number of cases which can be treated theoretically so as to derive formulæ for predicting the high frequency resistance, various attempts have been

made to devise means for measuring it in certain cases. The principle upon which the measurement is made is as follows.

Suppose two identical wires are taken, and let the steady resistance of each of them be R ohms. Let an alternating current A_1 having a frequency n be sent through one of the wires, and let R' be the high frequency resistance corresponding to this frequency; then the rate at which energy is dissipated as heat in this wire will be measured by A_1^2R'. Then let a steady unidirectional current of A amperes be passed through the other wire. The rate of dissipation of energy will be A^2R.

Suppose this steady current A is adjusted to have such a value that the rate at which heat is evolved in the two wires is the same, then we shall have

$$A^2R = A_1^2R',$$

or

$$\frac{R'}{R} = \frac{A^2}{A_1^2}. \quad . \quad . \quad . \quad (26)$$

Hence if we can determine when this equality exists and measure the alternating and direct currents, we can determine experimentally the ratio R'/R. This can be done by placing the two wires in two glass tubes which are connected together and form the two bulbs of a differential air thermometer. We can then determine when the rate of production of heat is equal in the two wires by adjusting the continuous current so that the air pressure in the two bulbs remains the same.

For full details as to the apparatus and mode of using it the reader must be referred to the author's book, *The Principles of Electric Wave Telegraphy and Telephony*, 3rd ed. chap. ii.

Experiments made with this apparatus confirm the truth of the formulæ already given for the high frequency resistance of straight wires and of spiral wires.

5. Inductance of Wires and Circuits

If a current i is flowing through a circuit at any instant there is energy associated with the circuit which is proportional to half the square of the current and to a constant L, called the *inductance* of the circuit. Hence the energy is measured by $\frac{1}{2}Li^2$. If the current changes, the rate at which this energy varies with the current is called the *electrokinetic momentum*. It is therefore measured by Li. The rate at which the kinetic momentum varies with the time is the self-

induced electromotive force in the circuit. It is therefore expressed by $L\frac{di}{dt}$.

Hence by Faraday's law the electrokinetic momentum is the number of lines of magnetic flux due to the current which are self-linked with the circuit.

We can therefore measure the inductance if we know the value of this self-induced electromotive force and the time rate of change of the current.

Definition.—If a current is varying, *i.e.* increasing or diminishing in a circuit of 1 turn at the rate of 1 *ampere* per second, and if the self-induced electromotive force in the circuit is then 1 *volt*, the circuit is said to have an inductance of 1 *henry*.

Small inductances may be measured in *millihenrys* or *microhenrys*, and also in absolute C.G.S. electromagnetic units of which 10^9 make 1 henry. The dimensions of a current on the electromagnetic system of units are $M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$, and therefore the dimensions of the time rate of change of current are $M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}\mu^{-\frac{1}{2}}$. Also the dimensions of electromotive force in the same system are $M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}$.

Accordingly the dimensions of inductance which are those of an electromotive force divided by those of the time rate of change of a current are $L\mu$.

Therefore the absolute unit of inductance in the electromagnetic system of units is 1 centimetre, and

$$1 \text{ henry} = 10^9 \text{ centimetres.}$$

The inductance or self-inductance L of any circuit may be therefore measured by the number of lines of magnetic flux which are self-linked with the circuit when unit current flows through it.

The *mutual inductance* M of two circuits is similarly measured by the number of lines of magnetic flux which are common or co-linked with the two circuits when unit current flows in both of them.

The inductance, whether self or mutual, of a circuit can be calculated by the aid of Neumann's formula.

Let ds be an element of length of one circuit and ds' that of another, and let θ be the angle between these elements. Let r be the distance of these elements, then

$$M = \int\int \frac{ds\, ds'}{r} \cos\theta \,. \quad . \quad . \quad (27)$$

If ds and ds' are elements of length in the same circuit, then the above formula gives the self-inductance.

If L_1 is the inductance of one circuit, and L_2 of another circuit, and if these circuits are in presence of each other and have a mutual inductance M, then the energy associated with the circuits when a current I_1 flows in the first and a current I_2 in the second is

$$W = \tfrac{1}{2}L_1I_1^2 + MI_1I_2 + \tfrac{1}{2}L_2I_2^2. \quad . \quad (28)$$

If H is the magnetic field at any point in the neighbourhood of a circuit conveying a current, and if we take any small area $\delta x \delta y$ in the field normal to the magnetic force H at the centre of this element of area, then the expression $\mu H \delta x \delta y$ is the number of lines of force passing through that area.

If this expression is integrated throughout the field both without and within the conductor, we obtain an expression for the whole magnetic flux linked with the circuit. If I is the current in the circuit, and L is its inductance, then the above integral is equal to LI. Hence the inductance L is the self-linkage of the circuit for unit current. In the same way the mutual inductance of two circuits is the common linkage of flux per unit current. An example will make this clearer.

The magnetic force due to an element of length ds of a circuit conveying a current i at a point at a distance r is by Ampère's law equal to $\frac{ids}{r^2}$ $sin\ \theta$, where θ is the angle between the direction of r and of ds.

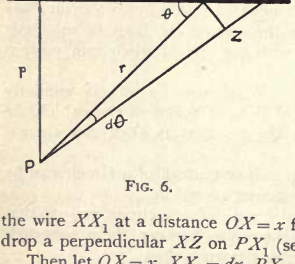


FIG. 6.

Consider then the case of a straight infinitely long wire. Take any point O on it as origin. Draw a perpendicular OP at that point. Take any element of length in the wire XX_1 at a distance $OX = x$ from the origin. From X drop a perpendicular XZ on PX_1 (see Fig. 1).

Then let $OX = x$, $XX_1 = dx$, $PX = r$, $OP = \rho$, and $OXP = \theta$. Then $sin\ \theta = \frac{\rho}{r}$, and if XX_1 is small the angle $XPX_1 = d\theta$.

Then $XZ = rd\theta$ and $\frac{XZ}{XX_1} = sin\ \theta = \frac{p}{r}$.

Accordingly we have

$$sin\ \theta = \frac{rd\theta}{dx} = \frac{p}{r},$$

and

$$\frac{dx}{r^2} = \frac{d\theta}{p}.$$

If i is the current in the wire, then the magnetic force at P due to the element dx is

$$\frac{idx}{r^2} sin\ \theta = \frac{i\ sin\ \theta d\theta}{p},$$

and the magnetic force of the whole infinite wire at the point P is

$$H = \frac{2i}{p} \int_0^{\frac{\pi}{2}} sin\ \theta d\theta = \frac{2i}{p} \quad . \quad . \quad (29)$$

It is therefore inversely proportional to the length of the perpendicular OP.

Consider then an element of area at the point P, which shall be a small rectangle of unit length parallel to the wire and width dp perpendicular to it. Then if the magnetic permeability of the medium around the wire is unity, the magnetic flux through the elements is $\frac{2idp}{p}$.

The integral of this is $2i\ log\ p$.

Hence the total flux surrounding this wire per unit of current and taken for a length l of the wire is

$$2l\ log\ p. \quad . \quad . \quad . \quad (30)$$

We can then apply this expression to the calculation of the inductance of a pair of parallel straight wires each of length l, and with centres at a distance of D centimetres, each wire having a diameter of d cms. We have then to calculate the value of the expression (30) between the limits $p = D$ and $p = \frac{d}{2}$, which gives $2l\ log \frac{2D}{d}$. This, however, has to be doubled, because the flux due to each wire separately is added together in the space between. Accordingly we have

$$L = 4l\ log \frac{2D}{d}. \quad . \quad . \quad . \quad (31)$$

for the inductance of a length l of a pair of parallel wires, one being the lead and the other the return. This formula is not quite accurate because we have neglected the flux inside the wires in obtaining it. Nevertheless it is a sufficiently near approximation for all cases in which the diameter d of the wire is small compared with the distance D apart.

The logarithm is Napierian. If it is desired to use ordinary logarithms, then it must be put in the form

$$L = 4l \log_{10} \frac{2D}{d} \times 2.3026. \quad . \quad . \quad (32)$$

For example.—Calculate the inductance of one mile run of a pair of parallel telephone wires each No. 10 S.W.G. and carried 2 feet 6 inches apart upon the posts.

Answer.—No. 10 S.W.G. wire has a diameter of 0.128 inch = 3.25 mm. Hence $d = 0.325$ cm. Also 1 foot = 30.48 cms. Therefore 2 feet 6 inches = 76.2 cms. Also 1 mile = 160933 cms. Accordingly

$$\frac{2D}{d} = \frac{152.4}{0.325} = 469.$$

$$\log_{10} 469 = 2.67117$$

and $$\log_e 469 = 2.3026 \times 2.67117 = 6.150$$

and $$4l = 64373.2 \text{ cms.}$$

Hence $$L = 643732 \times 6.150 = 3958951.8 \text{ cms.}$$

or $$L = 0.003959 \text{ henry}$$
$$= 3.959 \text{ millihenrys.}$$

The above formula cannot be applied to a pair of wires of which the length l is small compared with the distance between them, as it takes no account of the distribution of the magnetic flux in curved lines near the ends of the wires or loop.

An important formula is that for the *mutual inductance of a pair of parallel wires* each of length l placed at a distance D cms. apart, the wires having a diameter very small compared with their distance apart.

In this case we apply Neumann's formula.

Let the origin be taken at one end of the wires. Consider an element in one wire of length da at a distance a from one end of that wire; and also an element of length db in the other wire at a distance b from the corresponding end.

Then the distance apart of these elements is $\sqrt{(a-b)^2+D^2}$. Hence the mutual inductance M is given by

$$M=\int_0^l\int_0^l \frac{da.db}{\sqrt{(a-b)^2+D^2}} \quad . \quad . \quad (33)$$

We have then to integrate this expression between the limits o and l.

The general integral

$$\int \frac{dx}{\sqrt{x^2+c^2}} = log\ (x+\sqrt{x^2+c^2}).$$

Hence integrating (33) with respect to a alone between limits o and l we have

$$M=\int_0^l log\left\{\frac{\sqrt{(l-b)^2+D^2}+(l-b)}{\sqrt{b^2+D^2}-b}\right\}db.$$

Integrating again with respect to b we have

$$M=2l\ log\ \frac{\sqrt{(l^2+D^2)}+l}{D}-2\left\{\sqrt{l^2+D^2}-D\right\}.$$

If D is small compared with l this becomes

$$M=2l\left\{log\left(\frac{2l}{D}+\frac{D}{2l}\right)-\left(1-\frac{D}{l}+\frac{D^2}{2l^2}\right)\right\}.$$

If we can neglect D/l and higher powers this becomes

$$M=2l\left\{log\ \frac{2l}{D}-1\right\}. \quad . \quad . \quad (34)$$

The logarithm in (34) is Napierian, and if we wish to calculate with ordinary logarithms then we must write

$$M=2l\left\{2.3026\ log_{10}\ \frac{2l}{D}-1\right\}. \quad . \quad (35)$$

In the above (35) we must insert l and D in centimetres, and the result gives us M in centimetres or absolute E.M. measure.

To reduce to henrys divide by 10^9, and to reduce to millihenrys divide by 10^6.

For example.—What is the mutual inductance of two telephone wires, each 1 mile long and 5 feet apart?

Answer.— 5 feet = 152.4 cms. = D.
1 mile = 160933 cms. = l.

Therefore $\frac{2l}{D} = 2112$, and $log_{10}\ 2112 = 3.32469$.

Hence

$$M = 321866 \times (2.3026 \times 3.32469 - 1)$$
$$= 2,142,340 \text{ cms.} = 0.002142 \text{ millihenrys.}$$

We can make use of formula (34) to calculate the *self-inductance of a single straight wire*, such as an antenna wire.

If we have a current flowing in a wire of any section, we may regard that current as made up of a number of filamentary currents. Each pair of these filamentary currents has mutual potential energy, and the total potential energy of the whole current is the sum of the potential energies of all the pairs of circuits into which it can be divided.

Now formula (34) can be written

$$M = 2l\ (log\ 2l - 1 - log\ D)$$
$$= (2l\ log\ 2l - 2l) - 2l\ log\ D$$
$$= C - 2l\ log\ D. \quad . \quad . \quad . \quad . \quad (36)$$

Since l is constant, $2l\ log\ 2l - 2l$ can be written C, and is a constant.

Hence if we regard the current in a wire as made up of $2n$ filamentary currents we can divide these into n pairs, and if we suppose each filament carries unit current then the potential energy of the whole current is represented by the sum of all the potential energies of these n pairs of currents.

Let D_1, D_2, D_3, etc. be the distances of all the possible pairs of filaments, then the total potential energy is given by

$$nM = nC - 2l\ (log\ D_1 + log\ D_2 + \text{etc.} \ . \ . \ . \ + log\ D_n)$$
$$= nC - 2l\ log\ (D_1 D_2 D_3 \ . \ . \ . \ D_n). \quad . \quad . \quad (37)$$

Next let us suppose that the n pairs of filaments are arranged in two parallel close bundles, each containing n filaments and the centres of the bundles at a distance D.

Then the mutual induction of these parallel circuits is

$$nM = nC - 2ln\ log\ D. \quad . \quad . \quad (38)$$

Equating (37) and (38) we have

$$n\ log\ D = log\ (D_1 D_2 D_3 \ . \ . \ . \ D_n),$$

or

$$D = (D_1 D_2 D_3 \ . \ . \ . \ D_n)^{\frac{1}{n}}. \quad . \quad . \quad (39)$$

This value D is called by Maxwell the *Geometric Mean Distance* (G.M.D) of all the filaments as originally distributed. It can be calculated when we know the law of distribution over the cross section of the conductor.

Thus, if we suppose the current on a wire to be wholly on the surface the filaments are then arranged round the surface of a cylinder, say, of circular section. To find D we have then to calculate the G.M.D. of all points on the perimeter of a circle. It can be shown that this is equal to the radius of the circle.

If, on the other hand, the current on the wire is uniformly distributed over the cross section of the wire, then if that section is, say, a circle, the G.M.D. of all the filaments is the G.M.D. of all points on a circular area. It can be shown that this is equal to the radius divided by $\epsilon^{\frac{1}{4}}$ where ϵ is the base of the Napierian logarithms.

(See Clerk-Maxwell's *Treatise on Electricity and Magnetism*, 2nd ed. vol. ii. p. 298.)

Hence to obtain the inductance of a circuit we have the following rule:

Find the G.M.D. of the distribution of the filaments into which the current may be divided, and calculate the mutual inductance of two parallel circuits at that distance. This will give the self-inductance of the circuit.

Example.—Find the inductance of a straight circular-sectioned wire of length l and diameter d for high frequency currents.

Answer.—Since the currents are high frequency the current will be entirely on the surface of the wire, and we must assume the filamentary currents to be arranged round a circular cylinder. The G.M.D. of all points on a circle from one another is equal to the radius of the circle. Hence we have to calculate the mutual inductance of two straight wires each of length l at a distance apart equal to $\frac{d}{2}$, where d is the diameter of the wire. By (35) this is

$$M = 2l\left(2.3026\, log_{10}\frac{2l}{D} - 1\right),$$

and we have to substitute for D the value $\frac{d}{2}$. Hence we obtain for this straight wire of length l and diameter d an inductance

$$L = 2l\left(2.3026\, log_{10}\frac{4l}{d} - 1\right), \quad . \quad . \quad (40)$$

or

$$L = 2l\left(log_{\epsilon}\frac{4l}{d} - 1\right). \quad . \quad . \quad (41)$$

Example.—Calculate the inductance of an antenna wire 200 feet long and 0.1 inch diameter.

Here $l = 200$ feet, $d = 0.1$ inch,

$$\therefore \frac{4l}{d} = \frac{4 \times 200 \times 12}{0.1} = 96000,$$

and $$2.3026 \log_{10} \frac{4l}{d} = 2.3026 \times 4.98227 = 11.472.$$

Hence, since 1 foot = 30.48 cms. and 200 feet = 6096 cms. we have

$$L = 12192\ (11.472 - 1) = 127674.6 \text{ cms.}$$

The inductance is therefore 127674.6 absolute E.M. units or 0.127 millihenry.

Another important case is that of the mutual inductance of a pair of equal rectangular circuits with planes parallel and sides respectively equal to A and B centimetres, the distance apart of their planes being b.

The proof of the formula is rather long. The reader must be referred to the author's book, *The Principles of Electric Wave Telegraphy and Telephony*, 3rd ed. chap. ii. § 3, for the proof. The result is

$$M = 4\left\{ A \log \frac{A + \sqrt{A^2 + b^2}\sqrt{B^2 + b^2}}{(A + \sqrt{A^2 + B^2 + b^2})b} + B \log \frac{B + \sqrt{A^2 + b^2}\sqrt{B^2 + b^2}}{(B + \sqrt{A^2 + B^2 + b^2})b} \right.$$
$$\left. + 2\sqrt{A^2 + B^2 + b^2} - 2\sqrt{B^2 + b^2} - 2\sqrt{A^2 + b^2} + 2b \right\}. \quad (42)$$

We can apply this last formula to calculate the inductance of a rectangular circuit made of wire of a diameter d, the currents being high frequency currents, and therefore on the surface. We have then to substitute $\frac{d}{2}$ for b in formula (42), and if we can consider that d is small compared with the length A or B, we have the following formula for the inductance of the rectangular circuit when D is written for $\sqrt{A^2 + B^2}$.

$$L = 4\left\{ (A + B) \log \frac{4AB}{d} - A \log (A + D) \right.$$
$$\left. - B \log (B + D) - 2\,(A + B - D) \right\}. \quad (43)$$

The logarithms are Napierian and the value L is given in centimetres.

If we use ordinary logarithms then it must be remembered

that to obtain the Napierian logarithm we multiply the ordinary logarithm by 2.3026.

Hence (43) in terms of ordinary logarithms is

$$L = 9.2104\left\{(A+B)\log_{10}\frac{4AB}{d} - A\log_{10}(A+D) - B\log_{10}(B+D) - \frac{A+B-D}{1.1513}\right\}. \quad (44)$$

If in the above formula we put $A=B$ we obtain an expression for the inductance of a *square circuit*. It can be put in the form

$$L = 8A\left\{\log_e\frac{4A}{d} - \log_e(1+\sqrt{2}) + \sqrt{2} - 2\right\}. \quad (45)$$

The logarithms are Napierian and A stands for the length of the side of the square reckoned in centimetres.

A more convenient form is as follows.

Let l be the perimeter of the square in centimetres or sum of the four sides, then (45) can be put in the form

$$\left.\begin{aligned} L &= 2l\left(\log_e\frac{4l}{d} - 2.853\right) \\ \text{or} \quad L &= 2l\left(2.3026\log_{10}\frac{4l}{d} - 2.853\right) \end{aligned}\right\}. \quad (46)$$

If we compare this formula with (34) for the inductance of a straight wire of length l we see that they only differ in a numerical constant.

Also it can be shown that the inductance of a wire of length l bent into a circle, for currents of high frequency, is

$$\left.\begin{aligned} L &= 2l\left(\log_e\frac{4l}{d} - 2.45\right) \\ \text{or} \quad L &= 2l\left(2.3026\log_{10}\frac{4l}{d} - 2.45\right) \end{aligned}\right\}. \quad (47)$$

Hence if a wire of length l cms. and diameter d cms. is laid out straight or else bent into the form of a square or of a circle, in all cases we have the inductance given by the formula

$$\left.\begin{aligned} L &= 2l\left(\log_e\frac{4l}{d} - C\right) \\ \text{or} \quad L &= 2l\left(2.3026\log_{10}\frac{4l}{d} - C\right) \end{aligned}\right\}. \quad (48)$$

where $C = 1$ for a straight wire,
$C = 2.45$ for a wire bent into a circle,
$C = 2.853$ for a wire bent into a square,

provided the current is a high frequency current.

If the current is a steady or low frequency current then the current is distributed uniformly over the cross section of the wire, and if the wire is circular then the G.M.D. of all filaments is equal to $\frac{d}{2}\epsilon^{-\frac{1}{4}}$, where ϵ is the base of the Napierian logarithms. Hence if we require the inductance L of a straight wire for steady currents we have to substitute for D in formula (34) the value $\frac{d}{2}\epsilon^{-\frac{1}{4}}$. This gives us

$$L = 2l\left\{log\,\frac{4l}{d} - \frac{3}{4}\right\}. \quad . \quad . \quad (49)$$

Hence the above is the formula to use for steady currents, and the formula

$$L = 2l\left\{log\,\frac{4l}{d} - 1\right\}$$

is the formula for high frequency currents in calculating the inductance of a straight wire. The logarithms here are Napierian. To obtain the Napierian logarithm multiply the ordinary logarithm by 2.3026.

Turning then to spiral or coiled wires we have the following useful formula given by Dr. A. Russell for the inductance of a spiral wire having N turns per centimetre, the mean diameter of the spiral being D centimetres and length l centimetres.

$$L = (\pi DN)^2 l\left\{1 - \frac{4}{3\pi}\frac{D}{l} + \frac{1}{8}\left(\frac{D}{l}\right)^2 - \frac{1}{64}\left(\frac{D}{l}\right)^4\right\}. \quad (50)$$

This gives the inductance in centimetres or absolute E.M. units of a spiral coil, and division by 1000 gives it in microhenrys, by 1,000,000 gives it in millihenrys, and by 10^9 in henrys.

Example.—A spiral has 4 turns of wire per centimetre of length wound on a straight round rod such that the mean diameter of the spiral is 5 centimetres and the length of the spiral is 1 metre. Calculate the inductance.

Answer.—$D = 5$ cms., $l = 100$ cms., $N = 4$.

Hence $\frac{D}{l} = \frac{1}{20}$, $\frac{D^2}{l^2} = \frac{1}{400}$, $\frac{D^4}{l^4} = \frac{1}{160{,}000}$.

We may therefore neglect the last term. Also $\pi DN = 3.1415 \times 20 = 62.83$ cms., and therefore

$$L = (62.83)^2 \times 100\left\{1 - \frac{4}{9.425}\frac{1}{400} + \frac{1}{3200}\right\}$$

$$= 394761\left\{1 - \frac{1}{942} + \frac{1}{3200}\right\},$$

or

$$L = 394340 \text{ cms. nearly}$$
$$= 0.394 \text{ millihenry.}$$

The above formula shows that if the *dimension ratio* $\frac{D}{l}$ is as small as $\frac{1}{20}$ then the inductance per centimetre of length of the spiral is quite nearly enough given by the square of the length of the wire wound on 1 centimetre of length of the spiral.

Another important set of formulæ concern the inductance of spiral coils formed of insulated wire closely wound into cylindrical coils of one or more layers in depth and a number of turns in each layer such that the axial length of the coil is not large compared with its mean radius.

Maxwell shows (see *Treatise on Electricity and Magnetism*, vol. ii. chap. xiv. § 706, 2nd ed.) that the inductance L of a coil of N turns having a rectangular section of radial depth d and axial breadth b and mean radius r is given by the formula

$$L = 4\pi r N^2\left\{log\frac{8r}{R} - 2\right\}, \quad . \quad . \quad (51)$$

when R is the geometric mean of distance between all possible pairs of elements into which we can divide the cross section. The expression for this G.M.D. is a long one (see *The Principles of Electric Wave Telegraphy and Telephony*, J. A. Fleming, 3rd ed. chap. ii. § 3).

This expression is symmetrical in b and d, and shows that a coil of N turns has the same inductance if the mean radius remains the same, no matter whether it comprises x layers of y turns per layer or y layers of x turns per layer.

Hence a coil of N turns consisting of 1 layer in the form of a short cylinder has the same inductance as a flat coil of N turns, wound as a sailor winds up a spare rope, provided that the mean radius is the same in both cases.

Many formulæ have been given for the inductance of cylindrical coils of one layer of which the length is not large

compared with the mean radius. One of the most useful is that of Stefan, which is as follows :—

$$L = 4\pi r N^2 \left\{ \left(1 + \frac{3b^2 + d^2}{96r^2} \right) log_e \frac{8r}{\sqrt{b^2 + d^2}} - C_1 + \frac{b^2}{16r^2} C_2 \right\}, \quad (52)$$

where b is the axial breadth and d the radial depth of the rectangular section of the coil and r is the mean radius and N is the number of turns.

The constants C_1 and C_2 are functions of the ratio $\frac{b}{d}$ or $\frac{d}{b}$, which have been tabulated as follows :—

$\frac{b}{d}$ or $\frac{d}{b}$	C_1.	C_2.	$\frac{b}{d}$ or $\frac{d}{b}$.	C_1.	C_2.
0.00	0.50000	0.1250	0.55	0.80815	0.3437
0.05	0.54899	0.1269	0.60	0.81823	0.3839
0.10	0.59243	0.1325	0.65	0.82648	0.4274
0.15	0.63102	0.1418	0.70	0.83311	0.4739
0.20	0.66520	0.1548	0.75	0.83831	0.5234
0.25	0.69532	0.1714	0.80	0.84225	0.5760
0.30	0.72172	0.1916	0.85	0.84509	0.6317
0.35	0.74469	0.2152	0.90	0.86697	0.6902
0.40	0.76454	0.2423	0.95	0.84801	0.7518
0.45	0.78154	0.2728	1.00	0.84834	0.8162
0.50	0.79600	0.3066			

The ratio $\frac{b}{d}$ or $\frac{d}{b}$ is the ratio of the breadth to the depth of the winding measured over the insulation. Thus if the coil is a single layer winding of 10 closely adjacent turns, then $\frac{b}{d} = 0.10$ and the constants C_1 and C_2 are respectively 0.59243 and 0.1325.

The above formula requires a correction when using ordinary round wire covered with insulation, because then the current does not fill up the whole of the area occupied by the insulated wire. It has to be increased by a term

$$\Delta L = 4\pi r N \left(log \frac{D}{d} + 0.13806 + C \right), \quad . \quad (53)$$

where d is the diameter of the bare wire and D the over all insulation of the covered wire, and C is a constant depending on the number of turns of the wire N, as follows :—

Turns.	Layers.	*C*.
2	1	.006528
3	1	.009045
4	2	.01691
4	1	.01035
8	2	.01335
10	1	.01276
20	1	.01357
16	4	.01512
100	10	.01713
400	20 × 20	.01764
1000	50 × 20	.01778

Stefan's formula (52) applies not only to cylindrical coils of one layer but to flat spirals of wire of one layer, wound as a sailor coils up a spare rope. As an illustration of the use of this formula of Stefan we take the following.

Example.—A flat spiral coil is made of 10 turns of insulated wire closely adjacent, the mean radius being 10 cms. The wire is round copper wire insulated up to a diameter of 5 mm., and the ratio of diameter over insulation to diameter of bare wire is 2.6. Find the inductance of this spiral.

Answer.—We have $\frac{D}{d} = 2.6$, $r = 10$, $b = 0.5$, $d = 5.0$, $b^2 + d^2 = 25.25$, $\sqrt{b^2 + d^2} = 5.025$, $8r = 80$, $\frac{8r}{\sqrt{b^2 + d^2}} = 15.92$, $log_e\ 15.92 = 2.7676$, $4\pi r N^2 = 12566$, $1 + \frac{3b^2 + d^2}{96r^2} = 1.0026$.

Then

$$L = 12566\ \{1.0026 \times 2.7676 - 0.59243 + 0.1325 \times 0.00016\}$$
$$= 27424 \text{ centimetres.}$$

Also the correction ΔL for insulation space as given by the formula (53) is

$$\Delta L = 1256.6\ \{log_e\ 2.6 + 0.13806 + 0.01276\}$$
$$= 1382.$$

Hence $L + \Delta L = 28806$.

The inductance of the above spiral is therefore 28806 centimetres or absolute E.M. units, or 28.806 microhenrys, or 0.0288 millihenrys.

When the axial length b of the windings in a single layer cylindrical coil is small compared with the mean radius, as it is when a few turns of wire are wound on a large cylinder of

short axial length, then the formula of Lord Rayleigh for the inductance gives a very accurate result. This formula is

$$L = 4\pi r N^2 \left\{ log \frac{8r}{b} - \frac{1}{2} + \frac{b^2}{32r^2} \left(log \frac{8r}{b} + \frac{1}{4} \right) \right\}. \quad (54)$$

In the above formula the logarithms are Napierian.

This formula can be applied to calculate the inductance of the single layer spirals of closely adjacent turns which are used as tuning coils and inductances in radiotelegraphy.

CHAPTER IV

HIGH FREQUENCY CURRENT AND VOLTAGE MEASUREMENT

1. The Root-Mean-Square Value of a Simple Periodic Current and of a Damped Oscillation

IF a current is a simple periodic current or persistent oscillation, then the root-mean-square (R.M.S.) value of it is the square root of the mean of the squares of the instantaneous values of the current taken at equidistant intervals of time throughout one complete period. If i is the instantaneous value of the currents then this R.M.S. value is expressed by the integral

$$\sqrt{\frac{1}{\pi}\int_0^{\pi} i^2 dt}. \quad . \quad . \quad . \quad (1)$$

If the current is a persistent oscillation or so-called undamped oscillation, then it can be represented as a sine-function of the time, that is, $i = I \sin pt$ where I is the maximum value of the current during the period and $p = 2\pi n$ where n is the frequency; then we have

$$\text{R.M.S. value} = \sqrt{\frac{1}{\pi} I^2 \int_0^{\pi} \sin^2 pt\, dt}, \quad . \quad (2)$$

and this definite integral is equal to $\frac{I}{\sqrt{2}} = \frac{I}{1.414} = 0.707\, I$.

We can denote the R.M.S. value by (I), and it is the value of the current which would be read on a hot wire ammeter through which the current was passing. Accordingly

$$\left.\begin{aligned} I &= 1.414\,(I) \\ \text{or} \quad (I) &= 0.707\, I \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

If the current is a damped oscillation, that is, dies away in each train, then it can be represented by a function of the form

$$i = I\epsilon^{-\alpha t} \sin pt \quad . \quad . \quad . \quad (4)$$

where α is the *damping exponent* and ϵ is the base of the Napierian logarithm. Hence

$$i^2 = I^2\epsilon^{-2\alpha t} \sin^2 pt. \quad . \quad . \quad . \quad (5)$$

If the oscillations in each train of oscillation die away, forming what is called a *damped* train, then the Napierian logarithm of the ratio of one oscillation to the next succeeding oscillation in the opposite direction is called the *logarithmic decrement per semi-period.*

Let δ denote this *logarithmic decrement* (*log. dec.*), or *decrement* as it is called for shortness, then if I_1, I_2, I_3, etc., are the amplitudes of the 1st, 2nd, 3rd, etc., oscillations, we have

$$\delta = log \frac{I_1}{I_2} = log \frac{I_2}{I_3} \text{ etc., } . \quad . \quad . \quad (6)$$

and

$$\epsilon^\delta = \frac{I_1}{I_2} = \frac{I_2}{I_3} \text{ etc.} \quad . \quad . \quad . \quad (7)$$

Hence ϵ^δ is a factor by which we have to multiply the 2nd oscillation to obtain the 1st, and so on.

Hence $\epsilon^{\delta/2}$ is the factor by which we have to multiply the 1st oscillation to obtain the amplitude at zero oscillation. Accordingly the expression for the value of the current at any time t reckoned from the instant of starting the oscillation is

$$i = I_1\epsilon^{\delta/2}\epsilon^{-\alpha t} \sin pt. \quad . \quad . \quad . \quad (8)$$

Hence to obtain the R.M.S. value of the current we have to find the mean value of $\epsilon^{-2\alpha t} \sin^2 pt$ between certain limits.

If there are N trains of oscillations per second, each dying away to zero in $\frac{1}{N}$ part of a second, we may consider that the mean-square value required is that of

$$J^2 = NI_1^2\epsilon^\delta \int_0^\infty \epsilon^{-2\alpha t} \sin^2 ptdt. \quad . \quad . \quad (9)$$

Now

$$\sin^2 pt = \frac{1 - \cos 2pt}{2}$$

and

$$\int \epsilon^{-2\alpha t} \cos 2ptdt = \epsilon^{-2\alpha t}\left\{\frac{2p \sin 2pt - 2\alpha \cos 2pt}{4(\alpha^2 + p^2)}\right\} \quad (10)$$

Hence

$$J^2 = NI_1^2\epsilon^\delta \frac{p^2}{4\alpha(\alpha^2 + p^2)}.$$

But

$$p = 2\pi n \text{ and } \alpha = 2n\delta.$$

Therefore $$J^2 = \frac{NI_1^2\epsilon^\delta}{8n\delta} \cdot \frac{1}{1+\left(\frac{\delta}{\pi}\right)^2}. \qquad (11)$$

(See *The Principles of Electric Wave Telegraphy and Telephony*, by J. A. Fleming, 3rd ed. chap. iii.)

Now δ is generally a small quantity compared with π, and accordingly we may say that for feebly damped oscillations we have

$$J^2 = \frac{N\epsilon}{8n\delta} I_1^2 \quad . \quad . \quad . \quad (12)$$

or $$J = \sqrt{\frac{N\epsilon^\delta}{8n\delta}} I_1. \quad . \quad . \quad . \quad (13)$$

This gives us a relation between the amplitude of the 1st oscillation of current, viz. I_1, and the R.M.S. value, viz. J, of the numerous trains of oscillations. This last is the value which would be read on a hot wire ammeter through which the oscillations are sent.

In the above formula (13) N is the number of trains of oscillations per second or the so-called *spark frequency*, and n is the frequency of the oscillations in the train, and δ is the decrement per semi-period. If δ is very small then $\epsilon^\delta = 1 + \delta$ nearly.

The formula (13) shows that the R.M.S. value of an oscillation may be very small compared with the first maximum oscillation.

Example.—Let there be 400 trains of damped oscillations per second. Let the frequency of the oscillations be one million; that is, each complete oscillation occupies one-millionth of a second. Let the decrement per semi-period be .01.

Find the ratio between the R.M.S. value of the oscillations and the value of the 1st maximum oscillation.

Answer. — We have $n = 10^6$, $\delta = 0.01$, and $N = 400$. Therefore ϵ^δ = nearly unity, and $\frac{N}{8n\delta} = \frac{400 \times 100}{8 \times 10^6} = \frac{1}{200}$, and $\sqrt{\frac{1}{200}} = \frac{1}{14.14}$.

Hence $$I_1 = 14.14\,J,$$

or the 1st oscillation has a maximum value 14.14 times the R.M.S. value.

The same formula (13) applies to the R.M.S. value of

the oscillatory potential between two points in terms of the maximum value of the first oscillation, and of the other constants. Thus if U is the R.M.S. value of the potential terminal difference of a condenser, and V_1 is the maximum value of the 1st oscillation, then for damped trains of train frequency N and oscillation frequency n and semi-period decrement δ we should have

$$U = \sqrt{\frac{N}{8n\delta}} V_1. \quad . \quad . \quad (14)$$

2. Hot Wire Ammeters for the Measurement of High Frequency Currents

Although high frequency currents and oscillations exercise a magnetising effect on iron, and also circuits conveying them have electrodynamic action on each other in virtue of the magnetic fields surrounding these circuits, yet the only satisfactory method of measuring high frequency (H.F.) currents is by some form of thermal ammeter. Even then certain precautions are necessary. Ordinary hot wire ammeters generally contain a fixed shunt through which the major part of the current passes. The measuring part consists of a fine wire attached to the ends of the shunt, which is heated by a fraction of the current and by its expansion serves to measure it. This form of ammeter is totally unsuited for measuring high frequency currents, because the proportion of the current flowing through the shunt would change with the frequency and be in any case quite different from the proportion taken when direct currents were used.

The only satisfactory form of thermal ammeter is one such as that devised by the author in which the whole current passes through a number of fine wires in parallel, each of which is of so small a diameter that its resistance to a high frequency current, within the frequency used, is identical with its resistance to steady currents (see Chap. III.).

Then these wires must be so symmetrically arranged that they all become equally heated, and we can then test the temperature of one of them, either by its expansion or by means of a thermoelectric junction.

If these wires are so arranged in a case that they soon reach a final steady temperature when a current is passed through them, then we can conclude that heat is then produced at a certain rate by a current A, and if R is the resistance of

the wires the rate of heat production is A^2R. But since the fine wires have the same resistance for high frequency as for steady currents, we can conclude that an equality of current exists in the two cases when an equality of heat production exists.

Hence if we pass an oscillation through the wires and take a reading of the thermoelectric current produced in a junction held against one of them by means of a galvanometer, and if we repeat the experiment with a continuous current, we can then infer at once the ampere value of the oscillations.

In the Fleming High Frequency Ammeter there are two brass discs a few inches apart. These discs are carried on an insulating base of ebonite, and are connected to two terminals (see Fig. 7).

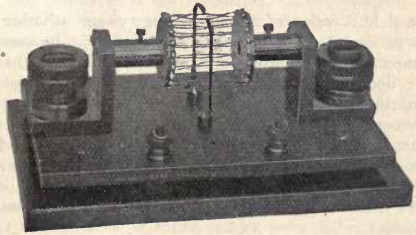


FIG. 7.—The Fleming High Frequency Hot Wire Ammeter.

These discs are connected by a certain number of fine copper wires (No. 36 S.W.G.) arranged like the staves of a barrel. To one of these wires a bismuth-iron or iron-constantan thermoelectric junction is attached, and the ends of this junction are connected to two separate terminals. To these last are joined a low resistance Paul single pivot sensitive ammeter for reading the thermocurrent.

If we pass through the fine wires a series of continuous currents of different values, the heat produced in the fine wires will create a thermocurrent, and the attached galvanometer will give a reading. We can easily plot a curve the abscissæ of which are proportional to the galvanometer scale readings, and the corresponding ordinates to the corresponding continuous currents which give that scale reading. If, then, we pass through the fine wires any electric oscillation, damped or

undamped, we shall obtain a certain deflection of the galvanometer, and from the plotted curve we can take off the corresponding current value in amperes of that oscillation.

By a suitable modification of the above ammeter we can read any high frequency current, large or small, the condition of accuracy being that the wires must be so fine that they have the same resistance for high frequency as for continuous currents.

For a description of various other forms of hot wire, or thermal ammeter for measuring high frequency currents, the reader can consult the author's book, *The Principles of Electric Wave Telegraphy and Telephony*, 3rd ed. chap. ii.

3. Measurement of High Frequency Voltage, Spark Voltages

The R.M.S. value of an alternating voltage, whether it is due to persistent or to damped oscillations, can be measured by means of an electrostatic voltmeter, provided that the voltage does not exceed a certain value. For this purpose we can use various forms of Kelvin multicellular or attracted disk voltmeter, or such a modification of Lord Kelvin's quadrant electrometer as the Dolezalek voltmeter, in which the "needle" consists of a dumb-bell-shaped strip of silvered paper suspended by a silvered quartz fibre. These instruments are described in most treatises on electrical measuring instruments.

Again, we can use some form of gold leaf electroscope in which the deflection of the strip of gold leaf can be measured on a scale. In all these cases we can only calibrate the electrometer, electroscope, or voltmeter if we have a battery of a known electromotive force.

We can accumulate a high known voltage, however, in the following manner. Let a number of condensers of equal known capacity be joined up in series. By means of a battery of voltaic or secondary cells of known terminal potential difference which can be measured by a potentiometer we can charge each condenser separately and then add the potentials by joining the condensers in series.

A simple form of switch can be constructed out of a block of paraffin wax, having small holes drilled in it which are filled with mercury. By this means a number of condensers of equal capacity can all be joined in parallel and each charged, say to 400 volts. Then by a simple change of

connections we can change the condensers from parallel to series and thus multiply the potential.

Thus if we have 20 condensers each of $\frac{1}{10}$ microfarad, we can charge them each to 440 volts from an electric supply circuit, and then by adding them in series obtain a known voltage of $20 \times 440 = 8800$ volts.

Or if we can obtain 1000 volts continuous current, which can be done by 500 small secondary cells, and if we have 20 condensers, we can obtain any voltage from 1000 to 20,000 and graduate an electrostatic voltmeter or electroscope so as to make it available for reading alternating current potentials.

It must, however, be noted that since every electrostatic voltmeter has some electrostatic capacity in itself, the reading it gives when its terminals are connected to two points is not necessarily the difference of potential between these points when the voltmeter is not connected. The voltmeter by its own capacity always more or less changes the potential difference of the points to which it is connected. It may lower it, and it may even increase it when alternating currents are being used by the action of electric resonance.

The maximum value of the potential difference between two points when it exceeds a certain limit can be ascertained by means of the length of the electric spark which this difference of potential can produce, provided certain precautions are taken as to the size and shape of the spark surfaces and also to prevent ultra-violet light, either from the sun or electric arcs or other sparks, from falling upon these surfaces. The determination of spark voltages or the potential difference required to make sparks of given length has been the subject of a large number of researches.

This spark voltage depends upon :—

1. The size and shape of the spark surfaces.
2. Upon the nature of the surfaces, viz. the metal of which they are made.
3. Upon the nature of the gas in which the discharge takes place.
4. Upon the pressure of that gas.
5. Upon the nature of the light which illuminates the spark surfaces.

The last condition is very important. Certain metals, such as zinc, aluminium, magnesium, and others, when bright and polished lose very quickly a negative charge of electricity if ultra-violet light falls upon them. Hence if two zinc balls are

connected to a high potential battery or coil and brought to sparking distance of each other, light from any source rich in violet rays will, when it falls on the negatively charged ball, greatly facilitate sparking.

Again, the sparking potential varies very greatly with the shape and size of the spark surfaces. It is not the same for spark balls of all sizes. It is therefore necessary to specify very precisely the size, nature, and shape of the spark surfaces.

We may for instance say that between brass balls 5 cm. in diameter placed in air at atmospheric pressure (760 mm.) a certain voltage between the balls would cause sparks to pass where the ball surfaces were at a certain distance.

The spark voltage increases with the pressure of the gas in which the balls are placed, and certain gases seem to have a restraining power, or, as it is called, high electric strength. It requires a certain electric force reckoned in so many volts per centimetre to break down the insulation of a gas and permit a spark to pass between electrodes, and this is best expressed in kilovolts per centimetre.

The true dielectric strength of air at atmospheric pressure lies between 38 and 39 kilovolts per centimetre. That is to say, between surfaces of a certain form a difference of potential of 39,000 volts is just sufficient to make a spark pass.

The dielectric strength can also be expressed in electrostatic units (E.S.): 1 electrostatic unit of electric force is 300 volts per centimetre.

Wolf has given a formula for the apparent dielectric strength of air at various pressures issuing as spark surface balls 10 cms. in diameter.

If E is the dielectric strength in electrostatic units, and P is the pressure in atmospheres, then

$$E = 107\ P + 39. \quad . \quad . \quad (15)$$

Hence according to this formula if $P = 1$, that is at ordinary atmospheric pressure, $E = 146$ E.S. units or $146 \times 300 = 43{,}800$ volts per centimetre.

The above formula is said to hold from 1 to 5 atmospheres.

Another empirical formula, by E. A. Watson, for the apparent dielectric strength of air between balls 2.54 cms. = 1 inch in diameter in kilovolts per centimetre, is

$$D = 25.6\ P + 20, \quad . \quad . \quad . \quad (16)$$

where P is the pressure in atmospheres.

Thus if $P = 1$, $D = 45.6$ or 45600 volts per centimetre.

The spark voltage or break-down voltage of air at various pressures is therefore very nearly a linear function of the pressure; or to put it in another way, the apparent dielectric strength of the air in kilovolts per cm. minus a certain constant is proportional to the pressure.

For the spark voltage in volts required to give a spark l *millimetres* in length between two brass balls, each 2.54 cms. (= 1 inch) in diameter, at ordinary atmospheric pressure and temperature, we have the following empirical formula:

$$V = 2121 + 3060\, l. \quad . \quad . \quad (17)$$

Thus to produce a spark of 10 millimetres between such balls it requires a voltage of $2121 + 3060 \times 10 = 32720$ volts.

The following tables for spark voltages between spark balls of various diameters are from the experimental results of Heydweiller, Algermissen, and Baille and Paschen.

TABLE SHOWING THE MEASUREMENT OF SPARK VOLTAGES FOR VARIOUS LENGTHS OF SPARK TAKEN BETWEEN SPARK BALLS OF VARIOUS DIAMETER AT NORMAL ATMOSPHERIC PRESSURE AND TEMPERATURE BY HEYDWEILLER.

v = spark voltage.
d = spark-ball diameter in centimetres.
l = spark length in millimetres.

d = 5.0 cms.		d = 2.0 cms.		d = 1.0 cm.		d = 0.5 cm.	
l in millimetres.	v in volts.	l in millimetres.	v in volts.	l in millimetres.	v in volts.	l in millimetres.	v in volts.
5	18,360	1	4,710	1	4,800	1	4,830
6	21,600	2	8,100	2	8,370	2	8,370
7	24,540	3	11,370	3	11,370	3	11,340
8	27,330	4	14,490	4	14,550	4	13,770
9	30,090	5	17,490	5	17,310	5	15,720
10	32,850	6	20,370	6	19,920	6	17,190
11	35,580	7	23,250	7	22,050	7	18,300
12	38,310	8	26,040	8	24,090	8	19,020
13	41,010	10	31,290	9	25,590	10	20,190
14	43,680	12	35,490	10	27,000	15	22,320
15	46,230	14	38,640				
16	48,660	16	41,280				

TABLE SHOWING THE SPARK VOLTAGE BETWEEN BRASS BALLS, 2 CMS. IN DIAMETER, FOR VARIOUS SPARK LENGTHS, BY HEYDWEILLER AND ALGERMISSEN.

Spark length in centimetres.	Spark voltage.	Spark length in centimetres.	Spark voltage.
0.1	4,700	2.7	54,900
0.2	8,100	2.8	55,800
0.3	11,400	2.9	56,700
0.4	14,500	3.0	57,500
0.5	17,500	3.1	58,300
0.6	20,400	3.2	59,000
0.7	23,250	3.3	59,700
0.8	26,100	3.4	60,400
0.9	28,800	3.5	61,100
1.0	31,300	3.6	61,800
1.1	33,300	3.7	62,400
1.2	35,500	3.8	63,000
1.3	37,200	3.9	63,600
1.4	38,700	4.0	64,200
1.5	40,300	4.1	64,800
1.6	41,300	4.2	65,400
1.7	43,200	4.3	66,000
1.8	44,700	4.4	66,600
1.9	46,100	4.5	67,200
2.0	47,400	4.6	67,800
2.1	48,600	4.7	68,300
2.2	49,800	4.8	68.800
2.3	51,000	4.9	69,300
2.4	52,000	5.0	69,800
2.5	53,000	5.1	70,300
2.6	54,000		

TABLE SHOWING THE SPARK VOLTAGE FOR VARIOUS SPARK LENGTHS AND SPARK-BALL DIAMETERS IN AIR AT ORDINARY PRESSURE, TAKEN FROM THE RESULTS OF BAILLE AND PASCHEN.

Spark length in millimetres.	Spark voltage for balls of diameter.			
	6 cms.	3 cms.	1 cm.	0.6 cm.
1	4,434	4,500	4,575	4,660
2	7,680	7,800	8,040	8,050
3	10,840	10,980	11,200	11,200
4	13,900	14,030	14,290	13,902
5	16,500	16,500	16,400	15,975
6	19,570	19,570	19,570	17,900
7	22,620	22,140	21,680	19,266
8	26,400	25,430	23,280	20,325
9	29,230	28,390	24,000	21,180
10	33,900	31,410	24,900	21,714

The degree to which the spark voltage varies with the size of the spark balls is shown by the figures given in the table below, in which the diameter of the spark balls in centimetres is marked below the name of the observer.

Spark length in air. d=	Spark voltage according to observations of— Bichat and Blondlot. 1 cm.	T. Gray. 70 cms.	T. W. Edmondson. 3 cms.	University College. 2.6 cms.	Mean results.
1 mm.	4,765	4,360	4,069	5,151	4,586
2 ,,	8,140	7,560	7,812	8,181	7,924
3 ,,	11,307	10,830	11,400	11,300	11,210
4 ,,	14,119	13,800	15,000	14,361	14,320
5 ,,	16,664	16,800	18,630	17,421	17,380
6 ,,	19,210	19,620	22,290	20,481	20,400

In some books tables of spark voltages are given between needle points. The term "needle" point is, however, a vague term, and may mean various degrees of sharpness or radius of curvature of the spark surfaces. Hence it is more indeterminate than when the spark surfaces are balls of given diameter of size about 1 inch or so. It is essential to notice that the spark voltage for a given spark length depends upon the size of the ball.

4. Relation of Current and Voltage in the Case of a Condenser charged with Alternating Current

A condenser of any form consists of a sheet or tube of insulating material, called the dielectric, bounded by two metallic surfaces, called the plates. The dielectric may be air or any other gas, or it may be oil or some insulating liquid, or it may be a solid insulator, such as glass, ebonite, or mica.

The condenser has a certain *capacity*, which means that if a difference of potential is made between the plates a certain quantity of electricity will be accumulated on them.

The capacity C of the condenser is measured by the quantity q required to charge it to unit difference of potential of the plates. Hence if q is the quantity in it when the potential difference (P.D.) of the plates is V we have

$$q = Cv. \quad . \quad . \quad . \quad . \quad (18)$$

If the condenser plates are connected by a short thick wire

and discharged, the current i at any instant flowing out is equal to the time rate of decrease of the charge, or

$$i = -\frac{dq}{dt} = -C\frac{dv}{dt}. \quad . \quad . \quad . \quad (19)$$

If the potential difference which is applied to the plates is a simple periodic function of the time, so that

$$v = V \sin pt, \quad . \quad . \quad . \quad (20)$$

where $p = 2\pi n$ and n is the frequency, then the current i flowing into the condenser at any instant is

$$i = CpV \cos pt. \quad . \quad . \quad . \quad (21)$$

Hence the maximum value of the current I must be expressed by

$$I = CVp. \quad . \quad . \quad . \quad (22)$$

If the current is measured in amperes and the capacity in microfarads we have

$$I = \frac{CV2\pi n}{10^6}. \quad . \quad . \quad . \quad (23)$$

where n is the frequency and $2\pi n = p = 6.28n$.

Example.—A condenser of 2 microfarads capacity has applied to its plates an alternating electromotive force of frequency 80. The root-mean-square value of this E.M.F. is 100 volts. Find the R.M.S. value of the currents flowing through the condenser.

Answer.—We have

$$C = \frac{2}{10^6} \text{ farads, } n = 80, \text{ and } \frac{V}{\sqrt{2}} = 100 \text{ volts.}$$

Hence $p = 2\pi n = 6.28 \times 80 = 502.4$, or say 500. Therefore

$$A = \frac{I}{\sqrt{2}} = \frac{2}{10^6} \times 500 \times 100 = 0.1 \text{ ampere.}$$

5. Current Flow through Circuits having Inductance, Resistance, and Capacity

If we apply a simple periodic electromotive force v represented at any instant by the function $v = V \sin pt$ to the ends of a circuit having resistance R and inductance L, then at that instant there will be a current i flowing through the circuit such that

$$v = V \sin pt = Ri + L\frac{di}{dt}. \quad . \quad . \quad (24)$$

For the impressed electromotive force v has to be equivalent to the sum of the effective electromotive force, which is equal to the product of the current and resistance, or to Ri and to the electromotive force of reactance which is due to the inductance, and this last is equal to the product of the inductance and the time rate of change of the current.

The solution of this equation (24) has been given in Chap. I. (see § 15, (125), (126)), viz.

$$i = \frac{V}{\sqrt{R^2 + p^2L^2}} \sin pt - \theta + \epsilon^{-\frac{R}{L}t}, \quad . \quad (25)$$

where $\tan \theta = \frac{Lp}{R}$.

The final term rapidly dies away as the time t increases, and we see that the maximum value of the current I is given by the expression

$$I = \frac{V}{\sqrt{R^2 + p^2L^2}}. \quad . \quad . \quad . \quad (26)$$

The denominator in this expression, viz. $\sqrt{R^2 + p^2L^2}$, is called the *impedance* of the inductive circuit, and may be graphically represented by the hypothenuse of a right angle-triangle, of which the horizontal side is the *resistance* R and the vertical side the *reactance* pL. In the case of such an inductive circuit the current, whether maximum value or R.M.S. value, is obtained by dividing the potential difference of the ends of the circuit by the impedance of the circuit.

Generally speaking, the *impedance* of a circuit for alternating currents is the ratio of the fall of potential down the circuit to the current through it, provided the circuit does not include any source of electromotive force other than the self-induction. This provides a means of measuring the inductance of the circuit. For if we measure with a hot wire ammeter the current through the circuit, and with a multicellular or electrostatic voltmeter the fall of potential down the circuit for any alternating current, and if we make the same measurement with a direct or unidirectional current we can find the value of the reactance, and hence of the inductance, if we know the frequency. Thus let I_a be the alternating current measured in amperes, and let V_a be the potential fall reckoned in volts, for an alternating current of frequency n. Also let

I_c and V_c be the same quantities for a continuous or direct current. Then we have

$$\frac{V_a}{I_a} = \sqrt{R^2 + p^2L^2} \quad \text{and} \quad \frac{V_c}{I_c} = R.$$

Hence

$$\left(\frac{V_a}{I_a}\right)^2 - \left(\frac{V_c}{I_c}\right)^2 = p^2L^2,$$

and

$$L = \frac{1}{2\pi n}\sqrt{\left(\frac{V_a}{I_a}\right)^2 - \left(\frac{V_c}{I_c}\right)^2} \quad . \quad . \quad (27)$$

and the inductance L can therefore be calculated.

Example.—It is found that to send a direct current of 5 amperes through a certain coil of wire requires an applied direct current voltage of 25 volts, but that to send an alternating current of 5 amperes through it requires an alternating voltage of 100 volts, the frequency being 80. Find the inductance of this coil.

Answer.—We have $I_c = I_a = 5$ amperes and $V_c = 25$ volts, $V_a = 100$ volts, and $n = 80$.

Hence $p = 2\pi n = 6.28 \times 80 = 502.4$, or say 500. Now

$\frac{V_c}{I_c} = 5$ and $\frac{V_a}{I_a} = 20$.

Hence

$$L = \frac{1}{500}\sqrt{400 - 25} = \frac{\sqrt{375}}{500} = 0.0385.$$

The inductance is therefore 0.0385 henry or 38.5 millihenrys.

Impedance and reactance are both measured in ohms, like resistance. Hence in this case the reactance is 19.35 ohms.

In the case of an inductive circuit traversed by an alternating current, the current considered as a vector is not in step with the terminal potential difference considered as a vector, but lags behind it by an angle, the tangent of which is the ratio of reactance pL to resistance R. The cosine of this angle, viz. $\frac{R}{\sqrt{R^2 + p^2L^2}}$, is called the *power factor* of the circuit.

If power is being supplied electrically to any device of any kind, the power given to it reckoned in watts is equal to the product of the current I flowing in it measured in amperes, and the terminal potential difference V measured in volts and the power factor. Since by equation (26) we have

$V = I\sqrt{R^2 + p^2L^2}$, and the power factor is $\dfrac{R}{\sqrt{R^2 + p^2L^2}} = \cos\theta$,

it follows that the power W given to the device is

$$W = IV\cos\theta = I^2R. \quad . \quad . \quad (28)$$

Hence the power factor may be defined as the quotient of the true power W in watts given to the circuit by the ampere-volts IV.

Thus the power factor of a closed iron circuit transformer taken on open secondary circuit would be about 0.7.

If instead of considering an inductive circuit we deal with a non-inductive resistance R, placed in series with a condenser of capacity C, we can apply to such a capacious circuit an alternating electromotive force, and this will create in it an alternating current.

Let v be the impressed E.M.F. at any instant and let $v = V\sin pt$, where $p = 2\pi n$ and n is the frequency, and hence V the maximum value of the E.M.F. Then if i is the current at that instant we shall have an equation of the following kind. The fall of potential down the non-inductive resistance R is Ri, and the fall in potential through the condenser is by equation (19)

$$i = C\frac{dv'}{dt}, \quad . \quad . \quad . \quad (29)$$

where v' is the instantaneous value of the potential difference of the condenser plates. Hence integrating we have

$$v' = \frac{1}{C}\int i\,dt. \quad . \quad . \quad . \quad (30)$$

Now the impressed E.M.F. v must be the sum of Ri and v'.

Hence
$$v = Ri + \frac{1}{C}\int i\,dt. \quad . \quad . \quad . \quad (31)$$

Differentiating this last with regard to t we obtain

$$\frac{di}{dt} + \frac{1}{RC} = \frac{1}{R}\frac{dv}{dt},$$

or
$$\frac{di}{dt} + \frac{1}{RC} = \frac{Vp}{R}\cos pt. \quad . \quad . \quad (32)$$

This differential equation is of the type

$$\frac{dy}{dx} + Py = Q \quad \text{(see Chap. I. § 15).}$$

and the solution of it is

$$y = \epsilon^{-\int P dx}\{\int Q\epsilon^{\int P dx} dx + c\} \quad . \quad . \quad (33)$$

where c is some constant of integration. (See equations 122-124, Chap. I.)

In this case $P = \frac{1}{RC}$, and $Q = \frac{Vp}{R} \cos pt$.

Therefore $\epsilon^{\int P dx} = \epsilon^{\frac{t}{RC}}$ and we have

$$i = \epsilon^{-\frac{t}{RC}} \int \left\{ \frac{Vp}{R} \epsilon^{\frac{t}{RC}} \cos pt dt + c \right\} . \quad . \quad (34)$$

$$= \epsilon^{-\frac{t}{RC}} \frac{Vp}{R} \int \epsilon^{\frac{t}{RC}} \cos pt dt + c\epsilon^{-\frac{t}{RC}} . \quad . \quad (35)$$

The integral of $\epsilon^{nx} \cos mx dx$ has been given in Chapter I. § 12. Hence it follows that the value of (35) is

$$i = \frac{VCp}{\sqrt{1 + C^2R^2p^2}} \sin \overline{pt + \theta} + c\epsilon^{-\frac{t}{RC}} \quad . \quad . \quad (36)$$

where $$\tan \theta = \frac{1}{CRp}.$$

The exponential term in (36) dies away rapidly, and hence we have

$$I = \frac{VCp}{\sqrt{1 + C^2R^2p^2}} \quad . \quad . \quad (37)$$

as an equation connecting the maximum value of the current I and that of the impressed E.M.F. V.

It is clear that if C is infinitely large, we have $I = \frac{V}{R}$ as it should be, and if $R = 0$, we have $I = VCp$ as in equation (22).

It is also clear that in this case the current is *in advance* of the impressed E.M.F. in phase by an angle θ such that $\tan \theta = \frac{1}{CRp}$, whereas in the case of the inductive circuit the current lags behind the impressed E.M.F. by an angle θ such that $\tan \theta = \frac{Lp}{R}$. It is easily seen that (37) can be written

$$I = \frac{V}{\sqrt{R^2 + \frac{1}{C^2p^2}}} \quad . \quad . \quad . \quad . \quad (38)$$

If we compare this equation with that giving the current in the inductive circuit, viz.:

$$I = \frac{V}{\sqrt{R^2 + p^2L^2}}, \quad . \quad . \quad . \quad (39)$$

we see that $\frac{1}{Cp}$ takes the place of Lp; in other words, the *capacity reactance* of a circuit containing a condenser of capacity C is $\frac{1}{Cp}$, whereas in the inductive circuit the reactance is Lp. The tangent of the phase angle between current and voltage is in both cases the ratio of reactance to resistance. These relations of voltage and current in the two kinds of circuit can be represented better by vector diagrams and complex quantities. Since alternating electromotive forces and currents have *phase* as well as *amplitude*, they can be represented by straight lines of certain lengths drawn at certain inclinations to the horizontal.

Let the black lines in the following diagrams marked V represent the maximum value of the terminal potential difference or *applied voltages* between the ends of some circuits having resistance R, and either inductance L or capacity C. Let $n = p/2\pi$ be the frequency. Let I be the maximum value of the current produced in the circuit. Then in each case the product RI is called the *effective voltage* in the circuit. In the case of the inductive circuit the product LpI is called the self-induced voltage or *reactance voltage*, and in the case of the condenser circuit $\frac{I}{Cp}$ is the reactance voltage.

In the diagrams the effective voltage is represented by a firm line marked RI, and the reactance voltage by a dotted line. Then for the inductive circuit the reactance voltage acts against the applied voltage, so that the latter has to be divisible vectorially into two components, one the effective voltage and another at right angles to it, viz. the reactance voltage. Hence the vector equation is

$$\mathbf{V} = \mathrm{R}\mathbf{I} + jpL\mathbf{I}. \quad . \quad . \quad . \quad (40)$$

The vector diagram is given in Fig. 8. Hence, taking sizes of vectors, we have

$$V = I\sqrt{R^2 + p^2L^2}. \quad . \quad . \quad . \quad (41)$$

For the circuit with capacity and resistance the reactance voltage acts with the impressed voltage, so that

$$\mathbf{V}+j\frac{\mathbf{I}}{Cp}=R\mathbf{I}. \quad . \quad . \quad . \quad (42)$$

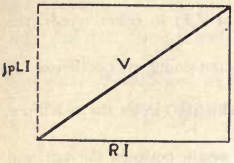


FIG. 8.

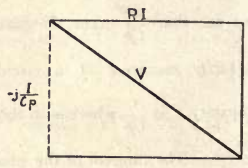


FIG. 9.

The vector diagram is given in Fig. 9, so that on taking sizes we have

$$V=I\sqrt{R^2+\frac{1}{C^2p^2}}. \quad . \quad . \quad (43)$$

Suppose then that the circuit comprises a condenser of capacity C in series with an inductive resistance of inductance

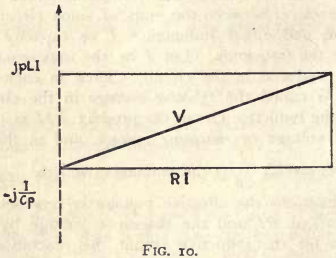


FIG. 10.

L and capacity C, it is obvious that the vector equation can be written

$$\mathbf{V}+j\frac{\mathbf{I}}{Cp}=R\mathbf{I}+jpL\mathbf{I}. \quad . \quad . \quad (44)$$

The vector diagram is given in Fig. 10. Hence the scalar or size equation is

$$V=I\sqrt{R^2+\left(Lp-\frac{1}{Cp}\right)^2}. \quad . \quad . \quad (45)$$

The quantity under the square root in (45) is the impedance of the compound circuit.

It is obvious that if the frequency n is such that $Lp = \frac{1}{Cp}$, or if $LCp^2 = 1$, or if $n = \frac{1}{2\pi\sqrt{LC}}$, then the whole circuit acts as if it were non-inductive. This is an important principle, as it shows that capacity can nullify inductance, or that capacity in a circuit acts like negative inductance.

Example.—A circuit has a resistance of 5 ohms and an inductance of 30 millihenrys. Suppose we apply to the ends of this circuit an alternating electromotive force of 100 volts (mean-square value), having a frequency 80, for which $p = 500$; find the current which will flow through this circuit and the capacity of the condenser which must be placed in series with it to first nullify the inductance.

Answer.—The impedance of the inductive circuit is $\sqrt{R^2 + p^2L^2}$. In this case $R = 5$ ohms, $L = 0.03$ henry, and $p = 500$. Hence the reactance $pL = 15$ ohms. Hence the impedance is $\sqrt{25 + 225} = \sqrt{250} = 15.81$ ohms. Therefore the current through the inductive circuit would be $\frac{100}{15.81} = 6.32$ amperes.

The capacity C required to nullify the inductance is such that $Cp = \frac{1}{Lp}$, or $C = \frac{1}{Lp^2}$. In this case $Lp = 15$ and $p = 500$, or $C = \frac{1}{7500}$ farads, or $\frac{10^6}{7500}$ microfarads. The required capacity is then $133\frac{1}{3}$ microfarads.

If this capacity is added in series with the coil of 5 ohms and 30 millihenrys, the circuit will behave to this alternating E.M.F. just as if it had only a resistance of 5 ohms and the current through it would rise to 20 amperes.

The frequency n, such that $LCp^2 = 1$ where $p = 2\pi n$, is called the *natural frequency* of the circuit whose inductance is L and capacity in series with it C.

6. Time Period of Electric Oscillations in Circuits having Inductance, Resistance, and Capacity

Let a condenser of capacity C be joined in series with an inductive circuit of inductance L and resistance R, which must be small. Let the circuit have a spark gap in it terminated by two spark balls (see Fig. 11). Such a circuit is called an oscillatory circuit.

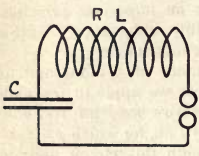


FIG. 11.

If a source of high electromotive force, such as an induction coil or transformer, is connected to the balls, then, if the balls are adjusted at a distance rather less than the spark length corresponding to the voltage of the supply, the condenser will be charged to this voltage, and a spark will then pass between the balls, momentarily connecting them by a conductor. The condenser will then discharge, but if the resistance is low this discharge will not be unidirectional but oscillatory, and consist of a rapid movement of electricity backwards and forwards in the circuit, the current gradually dying away.

These oscillations are isochronous, that is, are executed in equal periods of time with a periodic time T or frequency $\frac{1}{T}=n$, and they are damped with a certain damping factor $\epsilon^{-\delta}$ where δ is the logarithmic decrement per semi-period.

We can calculate this periodic time T or frequency n and decrement as follows:

If i is the current at any instant in the circuit, then $(R+jpL)i$ is the vector value of the fall in potential down the inductive resistance, and $-j\frac{i}{Cp}$ is the vector value of the fall in potential down the condenser. Hence $R+jpL+\frac{1}{jpC}$ is the vector value of the total drop in potential. But this is zero, because the circuit is closed and there is no impressed E.M.F. Hence we have

$$R+jpL+\frac{1}{jpC}=0, \quad . \quad . \quad . \quad (46)$$

or $$jpC\,(R+jpL)=-1,$$

or $$jpCR-p^2CL=-1. \quad . \quad . \quad (47)$$

But $-p^2=(jp)^2$ since $j=\sqrt{-1}$. Hence if we write z for jp, we can write (47) in the form

$$z^2+2\frac{R}{2L}z+\frac{R^2}{4L^2}=\frac{R^2}{4L^2}-\frac{1}{CL},$$

or

$$z+\frac{R}{2L}=\pm\sqrt{\frac{R^2}{4L^2}-\frac{1}{CL}}$$

$$jp=z=-\frac{R}{2L}\pm j\sqrt{\frac{1}{CL}-\frac{R^2}{4L^2}} \quad . \quad . \quad (48)$$

Accordingly, equating the vertical steps of this vector expression, we have

$$p=2\pi n=\sqrt{\frac{1}{CL}-\frac{R^2}{4L^2}} \quad . \quad . \quad (49)$$

or

$$n=\frac{1}{2\pi}\sqrt{\frac{1}{CL}-\frac{R^2}{4L^2}} \quad . \quad . \quad (50)$$

or

$$T=\frac{1}{n}=\frac{2\pi}{\sqrt{\frac{1}{CL}-\frac{R^2}{4L^2}}} \quad . \quad . \quad (51)$$

But if $\frac{R^2}{4L^2}$ is a small fraction, or small compared with $\frac{1}{CL}$, which it is in most cases, we have

$$n=\frac{1}{2\pi\sqrt{CL}} \quad . \quad . \quad . \quad (52)$$

or

$$T=2\pi\sqrt{CL}, \quad . \quad . \quad . \quad (53)$$

as expressions which give us the frequency n of the oscillations, or the time period T of the oscillations. It is seen that this frequency is the *natural frequency* of the condenser-inductance circuit. In the above expression the inductance and capacity must be measured in homologous units, that is, on units of the same kind.

Example.—A condenser of 1 microfarad is joined in series with a low resistance having an inductance of 10 millihenrys. Find the natural time period of oscillation of this circuit.

Answer.—10 millihenrys $=\frac{1}{100}$ of a henry.

1 microfarad $=\frac{1}{10^6}$ of a farad.

$2\pi=6.28$.

Hence the natural time period is

$$T = 6.28 \times \sqrt{\frac{1}{10^6} \times \frac{1}{100}} \text{ of a second}$$

$$= \frac{628}{1{,}000{,}000} \text{ of a second.}$$

This may be called 628 microseconds.

Example.—A charged Leyden jar having a capacity of $\frac{1}{400}$th of a microfarad has its coatings connected by a loop of copper wire 1 metre long and 4 millimetres in diameter bent into the form of a circle. Find the frequency of the oscillations set up in the work.

Answer.—From formula (47) in Chapter III. for the inductance of a circle of perimeter l we have

$$L = 2l\left\{2.3026\ log_{10}\frac{4l}{d} - 2.45\right\}$$

where d is the diameter of the wire.

In this case $l = 1$ metre $= 100$ cms., and $d = 4$ millimetres $= 0.4$ cm.

Hence $2l = 200$ and $\frac{4l}{d} = \frac{400}{0.4} = 1000$.

Also $log_{10}\ 1000 = 3$.

Hence $L = 200\ (3 \times 2.3026 - 2.45)$

$$= 892 \text{ cms.} = \frac{892}{10^9} \text{ henry.}$$

Also $C = \frac{1}{400}$ mfd. $= \frac{1}{4 \times 10^8}$ farad.

Therefore $$n = \frac{1}{6.28\sqrt{\frac{892}{10^9} \cdot \frac{1}{4 \times 10^8}}}$$

$$= \frac{10^8}{6.28\sqrt{22.3}} = \frac{100}{29.65} \times 10^6.$$

That is, about 3.3 millions per second.

The damping factor for such a circuit is ϵ^δ and $\delta = \frac{R}{4\pi L}$ where n is the frequency.

Hence, if R is large, the factor by which the amplitude of each oscillation has to be divided to obtain the amplitude of

the next oscillation in the opposite direction is also large and the oscillations decay away rapidly.

If there is a spark gap in the oscillatory circuit, this resistance R must include that of the spark itself.

If two circuits have the same natural time period of oscillation, they are said to be *in tune*.

The product $\sqrt{CL}$ is called the *oscillation constant* of the circuit.

CHAPTER V

CAPACITY MEASUREMENT AND PREDETERMINATION

1. Definition of Electric Capacity

THE capacity of a conductor is defined and measured by the quantity of electricity required to raise it to unit potential when all other neighbouring conductors are at zero potential.

If, under these conditions, the capacity of a conductor is denoted by C, and its potential by V, the quantity of electricity or charge in it is $Q = CV$.

Hence we can measure the capacity by the ratio of charge to potential.

In practical work capacity is measured either in *absolute electrostatic units* (E.S. units) or in *microfarads* or fractions of a microfarad (*mfd.*)

$$1 \text{ microfarad} = 9 \times 10^5 \text{ electrostatic units.}$$

A convenient unit for small capacities is the *micro-microfarad* (*m.mfd.*) or one-millionth part of a microfarad.

Hence

$$1 \text{ micro-microfarad} = 0.9 \text{ E.S. unit.}$$

The *farad* is 10^6 microfarads and 10^9 farads are 1 absolute electromagnetic unit (1 E.M. unit), and hence

$$9 \times 10^{20} \text{ E.S. units} = 1 \text{ E.M. unit of capacity.}$$

Since the *dimensions* of capacity in the E.S. system are those of a *length* multiplied by K, the dielectric constant, it is sometimes usual to state small capacities in *centimetres*, and then,

$$1 \text{ centimetre} = 1 \text{ E.S. unit.}$$

For radiotelegraphic work the most convenient unit is the micro-microfarad, which is one billionth part of a farad and might be called a *bifarad*.

2. Predetermination of Capacity in certain Cases

In certain simple cases it is possible to obtain a formula which enables us to calculate the capacity of conductors of certain shapes.

A Sphere.—If a charge Q is given to a sphere which is placed alone in space, then each element of that charge, viz. dQ, is at the same distance from the centre. Let R be the radius of the sphere, then $\frac{dQ}{R}$ is the potential of each element of charge at the centre of the sphere, and therefore $\frac{Q}{R}$ is the potential of the whole charge. But the potential of a conductor is the same at all points in it. Hence if V is this potential we have

$$V=\frac{Q}{R} \text{ and } \frac{Q}{V}=R\text{; but } \frac{Q}{V}=C$$

where C is the capacity of the sphere; therefore $C=R$, or the capacity in E.S. units is equal to the radius of the sphere in centimetres. The capacity of the sphere in microfarads is therefore

$$\frac{R}{9\times 10^5} \text{ mfds.} \quad . \quad . \quad . \quad (1)$$

Two Concentric Spheres.—If there be two concentric spheres, the radius of the inner being R_1 and the radius of the inside of the outer sphere being R_2, we can find the capacity as follows: Let a charge $+Q$ be placed on the surface of the inner sphere and let the outer sphere be put to earth. Then there is an induced charge $-Q$ on the inside of the outer sphere. Hence the potential at the centre of the inner sphere is $\frac{Q}{R_1}-\frac{Q}{R_2}$, and this is the potential of the inner sphere. Let this be called V. Then $Q/V=C$. Hence we have

$$C=\frac{R_1R_2}{R_2-R_1} \text{ (E.S. units).} \quad . \quad . \quad (2)$$

This gives the capacity in E.S. units if R_2 and R_1 are given in centimetres. To reduce it to microfarads divide by 9×10^5.

$$C=\frac{R_1R_2}{9\times 10^5(R_2-R_1)} \text{ (mfds.).} \quad . \quad (3)$$

A Long Vertical Wire.—Let a wire of diameter $2r$ and

length l be hung up in space a great distance away from the earth and other conductors.

Then we can calculate its capacity as follows: Suppose it divided up into thin cylindrical slices by planes perpendicular to its axis.

Let the thickness of one slice be dx, and let it be situated at a distance x from the centre of the wire taken as origin. Then the surface of the slice is $2\pi r dx$, and if the surface density of the charge is ρ, then the quantity of electricity on the curved surface of the slice is $2\pi r\rho dx$. The potential of this as the origin is dV, and

$$dV = \frac{2\pi\rho r dx}{\sqrt{r^2 + x^2}} \quad . \quad . \quad (4)$$

Hence, integrating from o to $\frac{l}{2}$ and doubling the result, we have

$$V = 2\int_0^{\frac{l}{2}} \frac{2\pi\rho r dx}{\sqrt{r^2 + x^2}} \quad . \quad . \quad (5)$$

Now by Chap. I. § 12

$$\int \frac{dx}{\sqrt{r^2 + x^2}} = log_e \left(x + \sqrt{r^2 + x^2} \right)$$

Hence $$V = 4\pi\rho r \left\{ log_e \left(\frac{l}{2} + \sqrt{r^2 + \frac{l^2}{4}} \right) - log_e r \right\}.$$

But $2\pi\rho r l$ is the whole charge Q on the wire, and $\frac{Q}{V} = C =$ the capacity. Therefore if r is small compared with $\frac{l}{2}$ we have

$$C = \frac{l}{2\ log_e \frac{l}{r}} \text{ (E.S. units)},$$

$$= \frac{l}{2\ log_e \frac{2l}{d}} \text{ (E.S. units)}. \quad . \quad (6)$$

where d is the diameter of the wire. In ordinary logarithms this becomes

$$C = \frac{l}{4.6052\ log_{10} \frac{2l}{d}} \text{ (E.S. units)}, \quad . \quad (7)$$

or
$$C = \frac{l}{9 \times 10^5 \times 4.6052 \log_{10} \frac{2l}{d}} \text{(mfds.)}. \quad . \quad (8)$$

Example.—An aerial wire is made of wire $\frac{1}{8}$th of an inch in diameter and 200 feet long. Find its capacity in microfarads.

Answer.— $l = 200$ feet $= 2400$ inches $= 6096$ cms.,

$d = \frac{1}{8}$th inch. Hence $\frac{2l}{d} = 38400$ $\log_{10} 38400 = 4.58433$.

Therefore
$$C = \frac{6096}{9 \times 10^5 \times 4.6052 \times 4.58433}$$
$$= \frac{61}{190{,}000} \text{ nearly},$$

or
$$C = \frac{1}{3000} \text{ (mfd.) nearly.}$$

If, however, we assume the earth to be a good conductor, and the lower end of the wire to be near to the earth, the actual capacity of the wire would be about double that given by the above formula.

Professor G. W. O. Howe has shown (see the *Wireless World*, Dec. 1914) that in the case of a wire of finite length, owing to the fact that there is a greater electric density at the ends than in the centre, the above formula for the potential, viz. $V = 4\pi\rho r \log_e \frac{l}{r}$, gives too large a value, and a more correct formula is

$$V = 4\pi\rho r \left(\log_e \frac{l}{r} - 0.309 \right).$$

Thus there is a correcting factor which is as follows for various values of $\frac{l}{r}$.

$\frac{l}{r}$.	$\log_e \frac{l}{r}$.	$\log_e \frac{l}{r} - 0.309$.	% diff.
200	5.30	4.99	6.18
600	6.38	6.07	5.1
2000	7.59	7.28	4.25
6000	8.68	8.37	3.7
20000	9.90	9.59	3.2
40000	10.59	10.28	3.0

A Pair of Parallel Wires.—If a pair of wires, each of diameter d and length l, are stretched parallel to each other with axes at a distance D apart, it can be proved that the capacity of these wires to each other is

$$C = \frac{l}{4 \log_e \frac{2D}{d}} \text{ (E.S. units)}, \quad . \quad . \quad (9)$$

or

$$C = \frac{l}{4 \times 9 \times 10^5 \times 2.3026 \log_{10} \frac{2D}{d}} \text{ (mfds.)}. \quad (10)$$

Since 1 mile = 160934.4 cms. the capacity per mile run of such a pair of wires is

$$C = \frac{0.0194}{\log_{10} \frac{2D}{d}} \text{ (mfds.)}. \quad (11)$$

The wires are supposed to be suspended in the air. This formula supplies the means of calculating the capacity of a pair of telephone wires, for instance.

A Horizontal Wire above a Conducting Earth.—A reduced case of the foregoing is that of a single horizontal wire of length l, suspended at a height h above a good conducting earth.

In this case there is an *electrical image* of the wire reflected in the earth, and the capacity of the single wire to the earth is double the capacity of a pair of such wires at a distance $2h$ from each other. Hence for this case

$$C = \frac{l}{2 \log_e \frac{4h}{d}} \text{ (E.S. units)} \quad . \quad . \quad (12)$$

or in microfarads it is

$$C = \frac{l}{2 \times 2.3026 \times 9 \times 10^5 \log_{10} \frac{4h}{d}} \text{ (mfds.)}, \quad (13)$$

or per mile it is

$$C = \frac{0.0388}{\log_{10} \frac{4h}{d}} \text{ (mfds.)}. \quad . \quad . \quad (14)$$

Two Concentric Cylinders.—If R_1 is the outer radius of the inner cylinder, and R_2 is the inner radius of the outer

cylinder, and l is the length, then if the space between the cylinders is filled with air the capacity is

$$C = \frac{l}{2 \log_e \frac{R_2}{R_1}} \text{ (E.S. units)}, \quad . \quad . \quad (15)$$

or in ordinary logarithms and microfarads

$$C = \frac{l}{4.6052 \log_{10} \frac{R_2}{R_1} \times 9 \times 10^5} \text{ (mfds.)}. \quad . \quad (16)$$

Since 1 mile = 160934.4 cms.

The capacity of a concentric cable per mile in microfarads is

$$C = \frac{0.0388\ K}{\log_{10} \frac{R_2}{R_1}} \text{ (mfds.)} \quad . \quad . \quad (17)$$

where K is the dielectric constant of the insulator.

This constant must in all formulæ be put in in the numerator if the dielectric is anything but air.

Example.—A paper insulated cable has a central conductor of which the outer diameter is 0.5 inch, and an outer concentric conductor of which the inner diameter is 1 inch. Find the capacity per mile in microfarads.

Answer.—For paper insulation the dielectric constant is about 1.25, and $R_2 = 1$, $R_1 = 0.5$, hence

$$\log_{10} \frac{R_2}{R_1} = \log_{10} 2 = .30103.$$

Therefore the capacity per mile is

$$\frac{0.0388 \times 1.25}{0.301} = \frac{485}{3010} = 0.162 \text{ mfd.}$$

Two Parallel Flat-plates.—Let there be two flat-plates each of area A parallel to each other. Let one be at potential V_1 and the other at potential V_2, and let Q be the quantity of electricity on each plate, the one being positively charged and the other negative. Let the distance between the plates be d. Then the electric force in the space between is $\frac{V_1 - V_2}{d} = E$.

If ρ is the surface density of the electricity on each plate supposed to be uniform, then $A\rho = Q$.

The relation between the surface density (ρ) and the

electric force E normal to the surface is expressed by Coulomb's law, viz.

$$E = 4\pi\rho. \quad . \quad . \quad . \quad (18)$$

Hence since $E = \frac{V_1 - V_2}{d}$, and since $\rho = \frac{Q}{A}$, we have

$$\frac{V_1 - V_2}{d} = \frac{4\pi Q}{A},$$

or

$$\frac{Q}{V_1 - V_2} = \frac{4}{4\pi d}, \quad . \quad . \quad . \quad (19)$$

but the capacity

$$C = \frac{Q}{V_1 - V_2}.$$

Hence we have

$$C = \frac{A}{4\pi d} \text{ (in E.S. units).} \quad . \quad . \quad (20)$$

Where A is the area of either plate measured in square centimetres, and d is the distance between the plates in centimetres.

If we reckon in microfarads the capacity is

$$C = \frac{A}{9 \times 10^5 \times 4\pi d} \text{ (mfds.).} \quad . \quad . \quad (21)$$

If the dielectric is other than air we must multiply by the dielectric constant K, and we have

$$C = \frac{KA}{36 \times 10^5 \times \pi d} \text{ (mfds.),} \quad . \quad . \quad (22)$$

$$= \frac{KA}{11309760\, d} \text{ (mfds.).} \quad . \quad . \quad (23)$$

A table of the values of the dielectric constants for various substances will be found at the end of this book (see Chap. XII.).

Example.—A condenser is formed of 19 square sheets of stout zinc, each 30 centimetres inside. These are placed parallel to each other, 0.5 centimetre apart, in a box full of paraffin oil. 10 alternate sheets are attached to one terminal, and 9 to the other. Find the capacity of the condenser so formed.

Answer.—The opposed surface is $18 \times 30 \times 30$ square centimetres, or $A = 16200$ sq. cms. The distance apart is 0.5 cm. or $d = 0.5$. The dielectric is paraffin, for which $K = 2$.

Hence the capacity in microfarads is

$$C = \frac{2 \times 16200}{11309400 \times \frac{1}{2}} = \frac{324}{56547} = 0.0057,$$

or about $\frac{1}{170}$th of a microfarad.

3. Experimental Determination of Capacities

In the majority of cases met with in practice the capacity cannot be pre-determined by calculation, but must be found experimentally. In radiotelegraphy we are especially concerned with the measurement of small capacities, that is to say, fractions of a microfarad—such, for instance, as the capacity of an aerial wire or wires, the capacity of a Leyden jar or jars, and of condensers which may be not a large fraction of a microfarad.

There are two methods which may be adopted. We may either compare the unknown capacity with the known capacity of some standard condenser, or we may make an absolute measurement, which is independent of any other standard of capacity. The latter method is an excellent one for small capacities, provided that the experimentalist possesses a suitable commutator.

The principle of the method is as follows: If a condenser is charged by a known electromotive force and then discharged through a galvanometer, and if this operation is repeated very quickly, say 100 times a second, the discharges run together into a practically continuous current and create a steady deflection on the galvanometer. If then the same source of electromotive is employed to create the same deflection in the galvanometer by inserting in series with it a suitable high resistance, and if need be shunting the galvanometer, we can calculate the capacity of the condenser in terms of a resistance and a time. Let C be the capacity of the condenser, and V the charging voltage, and let N be the number of times per second the condenser is discharged through the galvanometer. Then at each discharge a quantity of electricity CV passes, and hence the current through the galvanometer is NCV.

If then the resistance of the galvanometer is G, and if it is shunted by a shunt of resistance S, and if R is a resistance in series with the shunted galvanometer, then when the voltage V is applied, the current which flows through the galvanometer is

$$\frac{V}{R+\frac{GS}{G+S}} \times \frac{S}{G+S} = \frac{VS}{R(G+S)+GS}.$$

If this current gives the same deflection as the condenser discharges, we have

$$NCV = \frac{VS}{R(G+S)+GS},$$

or
$$C = \frac{S}{N\{R(G+S)+GS\}} \quad . \quad . \quad (24)$$

If R, G, and S are measured in ohms, and N is the number of discharges per second, then C will be given in farads, and must be multiplied by 10^6 to give microfarads. Hence

$$C = \frac{S10^6}{N\{R(G+S)+GS\}} \text{ (mfds.)}. \quad . \quad (25)$$

To conduct this measurement we require an appliance called a revolving commutator, which charges and discharges the condenser very regularly a known number of times per second.

An example of such an instrument, called the Fleming and Clinton commutator, will be found described in the author's book, *The Principles of Electric Wave Telegraphy and Telephony*, 3rd ed. chap. ii.

It consists of a metal barrel built up, like the commutator of an ordinary D.C. dynamo, of insulated slips of metal. Each alternate slip is connected to a metal strip on one side. There are three brushes which press against this commutator—one on each marginal strip and one on the divided central part. The barrel is rotated by an electric motor, and by means of an endless screw gearing a sound is made by a stroke on a gong every 100 revolutions. As the motor revolves the middle brush is connected alternately first to one and then to the other side brush. The middle brush is joined to one terminal of the condenser or object whose capacity is to be measured. The other side brushes are joined respectively to one terminal of the battery and of the galvanometer, whilst the remaining terminals of the battery, galvanometer, and condenser are joined together. By the above means it is quite easy to measure the capacity of an aerial wire, say 150 feet long, and having a capacity perhaps of $\frac{1}{3000}$th of a microfarad. It can also be applied to the measurement of the capacity of Leyden jars.

As the commutator itself possesses a certain capacity it is essential to make a blank correcting reading which is done by removing the condenser under test and taking a reading of the small residual deflection of the galvanometer. To calibrate the reading of the galvanometer we require a variable high resistance, say a megohm, divided into tenths and a variable shunt to the galvanometer. The voltage employed depends on the size of the capacity to be measured.

This commutator can also be used in another way. It can be applied to charge repeatedly a condenser and discharge it through a short thick wire. Such a combination of condenser and commutator is equivalent to a resistance equal to $\frac{10^6}{NC}$ ohms if C is the capacity in microfarads and N is the number of discharges per second.

For the combination acts like a resistance and passes through it a certain quantity of electricity per second. Hence the commutator and condenser can be treated like a resistance, and its value determined by insertion in the arm of a Wheatstone's bridge (see Chap. VI.).

If R and S are the resistances in the measuring arms of

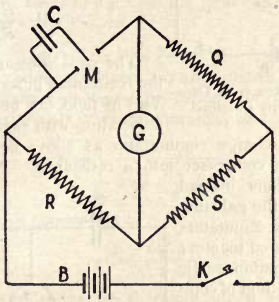


FIG. 12.—A Condenser charged and discharged by a Commutator measured as a Resistance.

the bridge (see Fig. 12), and if Q is the resistance balancing the condenser C and commutator M, and if B and G are the resistances of the battery and galvanometer, then it can be shown thus:

$$NC = \frac{S\left\{1 - \frac{S^2}{(Q+S+G)(R+B+S)}\right\}}{RQ\left\{1 + \frac{SB}{Q(R+B+S)}\right\}\left\{1 + \frac{SG}{R(Q+S+G)}\right\}}. \quad (26)$$

For the proof of this formula see J. A. Fleming, *A Handbook for the Electrical Laboratory and Testing Room*, vol. ii. chap. ii. § 6.

For the comparison of a condenser with a condenser of

known capacity de Sauty's method may be used. In this case a standard condenser having a capacity say of $\frac{1}{4}$ or $\frac{1}{2}$ a microfarad is connected up with the condenser of unknown capacity and with two variable resistances R_1 and R_2 so as to form a Wheatstone's bridge (see Fig. 13). The resistances are then varied until, when the battery key is pressed, the galvanometer G gives no "kick" or sudden deflection. When this is the case we have $C_1R_1 = C_2R_2$ or

$$\frac{C_1}{C_2} = \frac{R_2}{R_1} \quad . \quad (27)$$

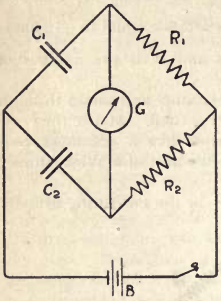


FIG. 13.—Comparison of Capacities by de Sauty's Method.

The capacities are inversely as the resistances in series with them. This method can be improved by associating with each condenser a charge and discharge commutator as above described so as to convert the condenser into a resistance. We then have steady deflections instead of "kicks" of the galvanometer. Both commutators must be connected together and driven simultaneously.

Another method of comparison is the "method of mixtures" originated by Lord Kelvin.

In this method a battery has its terminals connected by two variable resistances in series, and two condensers are arranged with a galvanometer as shown in Fig. 14. If the battery key is first closed the condensers will be charged with opposite potentials, and on putting down the galvanometer key there will generally be a "kick" on the galvanometer. If, however, the resistances R_1 and R_2 are varied but keeping the sum of R_1 and R_2 constant,

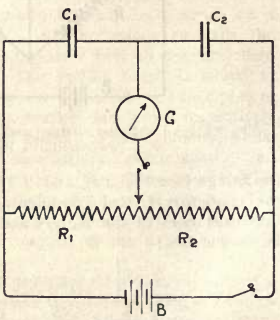


FIG. 14.—Comparison of Capacities by the Method of Mixtures.

then a point will be found at which the galvanometer gives no "kick" when the battery key is first closed and then the galvanometer key afterwards. When this is the case we have $C_1R_1 = C_2R_2$

or
$$\frac{C_1}{C_2} = \frac{R_2}{R_1}. \quad . \quad . \quad . \quad (28)$$

Various modifications of this method are in use, which will be found described in vol. ii. chap. ii. of J. A. Fleming's *Handbook for the Electrical Laboratory and Testing Room.*

4. Addition of Capacities

If we have several condensers of capacity C_1, C_2, C_3, etc., and we arrange them *in parallel*, the joint capacity is the sum of the separate capacities, or

$$C = C_1 + C_2 + C_3 + \text{etc.} \quad . \quad . \quad (29)$$

If, however, the condensers are arranged *in series*, then the resultant capacity C is found by taking the reciprocal of the sum of the reciprocals of each separately, or

$$C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \text{etc.}}. \quad . \quad . \quad (30)$$

Thus for two condensers

$$C = \frac{C_1C_2}{C_1 + C_2}.$$

Capacities in series are therefore added by the same rule as resistances in parallel, and capacities in parallel by the same rule as resistances in series.

Example.—Three condensers having respectively capacities of 1, 2, and 3 microfarads are joined in series. What is the resultant capacity?

Answer. $$C = \frac{1}{\frac{1}{1} + \frac{1}{2} + \frac{1}{3}} = \frac{6}{11} \text{ of a mfd.}$$

Example.—A capacity of $\frac{1}{500}$ microfarad placed in series with a condenser of 1 microfarad. What is the resultant capacity?

Answer. $$C = \frac{1}{\frac{1}{1} + 500} = \frac{1}{501} \text{ mfd.}$$

Hence the addition of a very large capacity in series with a very small one still leaves a small resultant capacity.

The above rules for the addition of capacities only hold good provided that the juxtaposition of the capacities does not affect the arrangement and number of lines of electric force proceeding from the conductors. Thus, if we have two flat plate condensers each consisting of a pair of metal plates very near to each other compared with their diameters, we can say that the capacity of the two condensers together is the sum of each separately. We could not, however, put two spheres in contact and say that the capacity of the two together is equal to the sum of their radii, that is of each taken alone, because the mere fact of putting them together alters the disposition of the lines of electric force, and therefore of the location of the electric charge on their surfaces. In the same manner a number of wires arranged as in a harp or fan antenna has a capacity far less than the sum of the individual wires of which it is composed.

Formulæ have been given by Professor Howe for the capacity of a number of associated wires such as are used in antenna construction. (See the *Wireless World*, December 1914 and January 1915.)

Thus if there are n parallel wires, each of semi-diameter r, equally spaced apart with a distance d between each, and each wire of length l, then Howe has given the following approximate formula for the capacity in electrostatic units of the whole of the wires taken together, viz.

$$C = \frac{nl}{2\left\{n \log_e \frac{l}{d} + \log_e \frac{d}{r} - B\right\}} \text{ (E.S. units)}, \qquad (31)$$

where B has the following values for various values of n.

No. of Wires $= n$.	$B =$	No. of Wires $= n$.	$B =$
2	0	8	6.40
3	0.46	9	8.06
4	1.24	10	9.80
5	2.26	11	11.65
6	3.48	12	13.58
7	4.85		

Professor Howe has given the following formula as a better approximation for the capacity of a flat or harp parallel wire

antenna consisting of n equal wires spaced apart by a distance d provided that the total length l is a large multiple of the total width nd:

$$C = \frac{nl}{2\left\{n\left(\log_e \frac{l}{d} - 0.309\right) + \log_e \frac{d}{r} - B\right\}} \quad . \quad (32)$$

The above is in electrostatic units, and is reduced to micro-microfarads by dividing by 0.9.

The constant B has the values in the table above given.

Where greater accuracy is required we must substitute for $\left(\log_e \frac{l}{d} - 0.309\right)$ in the above formulæ the longer expression

$$\left(\sinh^{-1} \frac{l}{d} - \sqrt{1 + \frac{d^2}{l^2}} + \frac{d}{l}\right). \quad . \quad . \quad (33)$$

In the paper above mentioned Professor Howe has applied the formula above given to the calculation of the capacity of a number of parallel wire antennæ.

All lengths l, d, and r must be in centimetres.

The upshot of the matter is that the capacity of a number of parallel wires, if they are at all close, is not nearly equal to the sum of the capacities of each wire taken alone. It is much less. Thus, in the case of some experiments tried by the author (see *Principles of Electric Wave Telegraphy and Telephony*, J. A. Fleming) eleven strips of iron, each 15 feet long, were hung up in a room and the capacity measured of all together. Taking the capacity of each strip alone as unity, the capacity of the eleven strips was only 5.9 when they were 12 inches apart, 4.0 when 6 inches, 2.99 when 3 inches, and 1.99 when in contact. When spaced very far apart the total capacity approximated to 11.

5. Dissipation of Energy in the Dielectrics of Condensers

When a condenser of capacity C is charged so that the terminal plates have a difference of potential V the energy stored up in the condenser is measured by $\frac{1}{2}CV^2$ or $\frac{1}{2}\frac{Q^2}{C}$ where Q is the charge on each plate.

If C is measured in farads and V in volts then the stored energy is in *joules*.

Example.—Twelve Leyden jars each of $\frac{1}{400}$th of a microfarad capacity are charged to a potential represented by a

7-millimetre spark between balls 2 inches in diameter. What is the energy stored up?

Answer.—A 7-millimetre spark represents a potential of 20,000 volts. The capacity of the Leyden jars is $\frac{12}{400}\frac{1}{10^6}=\frac{3}{10^8}$ farad. Hence the energy storage is equal to

$$\frac{1}{2}\frac{3}{10^8}(20{,}000)^2=6 \text{ joules.}$$

Since 1 joule is about $\frac{3}{4}$ of a foot-lb., this represents about 4.5 foot-lbs. of energy.

If a condenser is made with a perfect dielectric the energy obtained from it on discharge should be equal to that put into it on charge, and, moreover, this energy should go in and out instantaneously. As a matter of fact no real dielectric, except the æther as exhibited by a high vacuum, is perfect in this sense.

All solid dielectrics exhibit in some degree the phenomena of absorption and energy dissipation.

The property of electric absorption may be explained by analogy with the imperfect elasticity of beams under mechanical stress. If we support a beam at the ends and place a weight in the middle the beam suffers a *deflection*.

If the weight is put on and removed very soon the deflection produced will then almost entirely disappear, but if the weight is allowed to remain on for some time a slowly increasing additional deflection will be produced which will not all disappear on the removal of the weight. Part of it will disappear at once, part will slowly disappear, and part may never disappear, but the beam will take a permanent *set* if the material is imperfectly elastic.

These facts with regard to the mechanical strain of beams under mechanical stress have their exact analogues in the case of the *electric displacement*, or electric strain produced in dielectrics by *electric stress* or force. If a condenser has applied to it a voltage for a very short time, and if the voltage is then removed and the condenser short-circuited, or if the voltage is reversed in direction, the charge which comes out of the condenser is almost exactly equal to that going into it. If, however, the voltage is kept on for some time, then the charge slowly increases and does not all come out at once on discharge. Part of it comes out instantly and part more slowly, and if the dielectric has any true conductivity part never comes out at all. The part of the charge which comes

out slowly is called the *absorption.* If the voltage is an alternating current voltage it is found that there is a certain dissipation of energy in the condenser which exhibits itself as heat in the dielectric, which may be due to true conductivity in the dielectric or in part to more obscure effects generally called dielectric hysteresis. We can, however, represent all the energy loss in a condenser as if it were due to a certain resistance arranged in series with or in parallel with the true capacity. Such a leaky condenser is called an imperfect condenser and the energy loss due to it is called the *dielectric loss.*

If a condenser of capacity C, the dielectric of which is perfect, has an alternating voltage V of frequency $n=p/2\pi$ applied to it, and if there is a conductance S in parallel with the condenser, then the current I through this conductive capacity is expressed by the vector equation

$$\mathbf{I} = S\mathbf{V} + jpC\mathbf{V}$$
$$= (S + jpC)\mathbf{V}. \quad . \quad . \quad . \quad (34)$$

The quantity $\frac{\mathbf{I}}{S+jpC}$ or $(S+jpC)^{-1}$ is called the *vector impedance* of this leaky condenser and $S+jpC$ is called the vector admittance. If we scalarize equation (34) we have

$$I = V\sqrt{S^2 + p^2C^2} \quad . \quad . \quad (35)$$

and $\sqrt{S^2+p^2C}$ is called the admittance of the condenser.

If we put a resistance R in series with a perfect condenser of capacity C, and apply to the system an alternating voltage of frequency $n=p/2\pi$, then the vector equation for the current is

$$\mathbf{V} = R\mathbf{I} - \frac{j\mathbf{I}}{pC}$$
$$= \left(R - \frac{j}{pC}\right)\mathbf{I}. \quad . \quad . \quad (36)$$

The quantity $\left(R - \frac{j}{pC}\right)$ is called the vector impedance of the condenser with resistance in series with it. Hence scalarizing we have

$$V = I\sqrt{R^2 + \frac{1}{C^2p^2}} \quad . \quad . \quad (37)$$

and the quantity $\sqrt{R^2+\frac{1}{C^2p^2}}$ is called the *impedance* of the condenser in series with a resistance. The equation (37) is consistent with equation (37) in Chapter IV. Hence equating vector impedances in the two cases we have

$$\frac{1}{S+jpC}=R-\frac{j}{Cp}$$

or

$$\frac{S-jpC}{S^2+p^2C^2}=R-\frac{j}{Cp}.$$

Equating horizontal and vertical steps we have

$$\frac{S}{S^2+p^2C^2}=R$$

and

$$\frac{pC}{S^2+p^2C^2}=\frac{1}{Cp}.$$

Hence $$\frac{S}{C^2p^2}=R \text{ or } S=C^2p^2R. \quad . \quad . \quad (38)$$

We can therefore substitute for a given conductance S in parallel with the capacity C a resistance R in series with it provided $R=S\frac{1}{C^2p^2}$.

We may therefore regard a condenser made with an imperfect dielectric as equivalent to a condenser made with a perfect dielectric having in series with it a resistance. We cannot, however, represent all the phenomena of a condenser such as those called the residual charge effects by this simple arrangement, but the author has shown that all phenomena connected with condensers made with imperfect dielectrics can be represented by a collection of perfect condensers each having certain resistances in series with them, these series being arranged in parallel.

6. Measurements of Energy Losses in Condensers with Alternating Currents

The principles explained in the previous section can be applied in measuring the energy losses in condensers with alternating currents in the following manner.

Suppose a condenser given having a leaky or conductive dielectric. It is required to find the true capacity C and the true conductance S of the condenser for alternating currents

of a frequency $n = p/2\pi$, such frequency lying within the range for which a telephone gives a good audible note say between $n = 100$ and $n = 5000$. We join up the condenser, which can be represented as shunted by a resistance to denote its dielectric conductance, with three other non-leaky condensers which should preferably be air condensers of variable capacity. These condensers are arranged in the way shown in Fig. 15, forming what is called a Capacity Bridge. The condenser under test has a capacity C and a leakance or conductivity S.

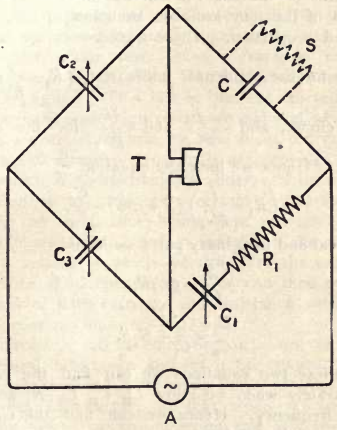


FIG. 15.—Capacity Bridge (Fleming and Dyke).

Opposite to this is placed an air condenser of capacity C_1, having a variable resistance R_1 placed in series with it.

Two other variable air condensers of capacities C_2 and C_3 are placed as shown in the diagram.

A high resistance telephone is placed as a bridge across one diagonal, and the other diagonal points must be supplied with alternating currents of a frequency say between 500 and 5000, the electromotive force being of pure sine wave form.

The particular instructions for obtaining this E.M.F. are contained in a paper by Dr. J. A. Fleming and Mr. G. B. Dyke published in the *Journal of the Institution of Electrical*

Engineers of London, 1912, vol. xlix. p. 323, entitled "On the Power Factor and Conductivity of Dielectrics when tested with Alternating Electric Currents of Telephonic Frequency at Various Temperatures."

The particular measurement made with this capacity bridge consists in adjusting the capacities of the three condensers C_1, C_2, C_3, and the resistance R_1 so that no sound is heard in the telephone when the alternating E.M.F. of frequency $n = p/2\pi$ is applied to the arrangement.

When this is the case it can be shown that the vector impedances of the four arms or branches of the circuit are in continued proportion. These impedances are respectively $(S+jpC)^{-1}$ for the condenser under test, $\left(R_1 - \frac{j}{pC_1}\right)$ for the balancing circuit, and $-\frac{j}{pC_2}$ and $-\frac{j}{pC_3}$ for the other two condensers. Hence we have the equation

$$\left(R_1 - \frac{j}{pC_1}\right)(S+jpC) = \frac{C_2}{C_3}.$$

Equating real and imaginary parts we have

$$\frac{S}{Cp} = pC_1R_1,$$

$$\frac{C_2}{C_3} = \frac{C}{C_1} + R_1S.$$

From these two equations we can find the values of S and C separately when we know C_1, C_2, C_3, R_1 and $p = 2\pi n$ where n = frequency. Hence we can find the conductance and capacity of the condenser.

From experiments made with the above method the author showed that the conductivity of a dielectric for alternating currents of frequencies between zero and 5000 increases as a linear function of the frequency, so that if s is the conductivity of the dielectric

$$s = a + bn,$$

where a and b are constants and n is the frequency.

For higher frequencies such as 0.5 to 1 or 2 millions, such as are used in radio-telegraphy, it has been shown by Dr. G. E. Bairsto that there is a maximum value for the conductivity corresponding to a certain high frequency special to each dielectric.

The results show that for all solid ordinary dielectrics such as glass, ebonite, gutta-percha, indiarubber, paper, celluloid, etc., the true conductivity of the dielectric for alternating currents is much greater than its true conductivity for direct currents.

For the purposes of radio-telegraphy we require simple methods of testing the energy losses in condensers for high frequency currents. The above-described method with the capacity bridge is not well adapted for such work, and a better method, making use of damped high frequency oscillations, is as follows :—

A variable capacity air condenser must be provided, the capacity of which can be made equal to the true capacity of the condenser under test. Also a variable non-inductive resistance must be provided, the inductance of which is perfectly negligible. This last is best constructed as follows. Two fine wires of some high resistance material, say constantan or eureka, are stretched side by side about 0.5 cm. apart on a long strip of ebonite. Terminals are provided at one end of each wire. A short-circuiting slider can be moved along the wires so as to intercept a certain length and vary the resistance, the inductance being kept extremely small. If this variable resistance is put in series with the air condenser we have a system of which we can alter the capacity C or the resistance R independently. We can then compare this combination of pure capacity and resistance with the actual leaky condenser as follows :—

Let a circuit be set up consisting of a coil of wire acting as a primary inductance and let it be connected in series with a condenser and with some form of quenched spark or dead beat discharger, so that when the condenser is charged by means of an induction coil or transformer and allowed to discharge itself across this spark gap, it produces a dead beat discharge or one consisting of a single oscillation or so.

Then set up near this primary inductance a secondary circuit which has in series with it either the condenser under test or else the air condenser and adjustable resistance above described.

Also put in series with this circuit some form of hot wire ammeter suitable for reading high frequency currents.

Again, a tertiary or *cymometer circuit* (see Chap. VII.) must be set up consisting of a coil of wire placed near to the secondary inductance and joined in series with a variable air condenser.

Then insert in the secondary circuit the condenser under test and make the following adjustments :—

Start the discharges in the primary circuit. This will set up in this circuit trains of oscillations each comprising very few oscillations or very highly damped. These will in turn create in the secondary circuit trains of oscillations having a large number of oscillations in each train or feebly damped trains. Alter the capacity or inductance either in the primary or secondary circuit until the current, as shown by the hot wire ammeter in the secondary circuit, has a maximum value.

Then in the same manner tune the tertiary circuit by altering its variable capacity until a hot wire ammeter inserted in the tertiary circuit shows a maximum current. We have then three circuits in tune with each other.

Next substitute in the secondary circuit for the condenser under test the air condenser with the variable resistance in series with it. Adjust the capacity of the condenser until the current in the secondary circuit is a maximum and also the current in the tertiary circuit a maximum. We then know that the capacity of the air condenser in the secondary circuit is equal to the true capacity of the condenser under test. Next adjust the variable resistance in the secondary circuit until the value of the current as shown by the hot wire ammeter is equal to its value when the condenser under test is in that circuit. Then we can take the value of this resistance as equal to the equivalent resistance of the condenser under test, and we can say that for any given mean-square value of the current through that condenser the product of this resistance and the mean-square current would give the rate of energy loss in watts in that condenser.

Any condenser may, therefore, as regards the energy dissipating power of its dielectric be looked upon as if it consisted of a pure capacity having in series with it a certain hypothetical resistance which is not constant but increases rapidly in value with the increase of the alternating current flowing through the condenser.

The loss in the dielectric can then be expressed by the product of this resistance and the mean-square value of the current flowing through the condenser. It can be stated as so many microwatts per cubic centimetre of dielectric for a given electric stress or volt fall per centimetre across the dielectric. The experiments above mentioned showed that the energy loss in air and in certain highly insulating and perfectly dry mineral oils is very small when these are used

as dielectrics in condensers when the electric force in the dielectric is small. If, however, the forces rise above 2 E.S. units, equal to 600 volts per centimetre, then even for these oils the dielectric loss may amount to several thousand microwatts per cubic centimetre.

On the other hand, in glass or ebonite the dielectric losses may be very large, amounting to several thousand microwatts per c.c. for electric forces not exceeding 1.5 E.S. units, whilst for electric forces of the order of 10 E.S. units the loss may be from $\frac{1}{4}$ to $\frac{1}{2}$ a watt per c.c. in certain kinds of glass.

These energy losses mount up with great rapidity as the electric force increases, just as the magnetic hysteresis loss in iron rises very rapidly with the flux density. Hence all large condensers should have dielectrics which are traversed only by electric forces as small as possible, which means that energy should be as far as possible stored by the use of a large capacity charged to a low voltage rather than by a small capacity charged to a high voltage.

Hence in any oscillation circuit the inductance should be kept small and the capacity large.

The subject of condenser losses is again mentioned in Chap. VII., under the heading of Decrement Measurements.

CHAPTER VI

BRIDGE METHODS OF MEASUREMENT OF RESISTANCE AND INDUCTANCE

1. Networks of Conductors, Bridges, and Potentiometers

If a number of points are joined by wires so as to form a set of closed circuits which are interconnected, this is called a network of conductors. Each closed circuit is called a *mesh*, and each mesh is bounded by a set of conductors. If an electromotive force is applied in one or more of those conductors we have currents set up in the conductors.

There are certain laws which control these currents and electromotive forces, as follows :—

1. At each junction point the algebraic sum of the currents must be zero, reckoning the current as positive when it flows to a junction and negative when it flows from it. This follows because there can be no accumulation of electricity at a junction.

2. If we pass round the boundary of each mesh and multiply the resistance of each conductor by the current in it, the sum of all those products is equal to the sum of all the electromotive forces in those circuits, reckoning the products and E.M.F.'s as positive when they are directed with the direction of travel and negative when against it.

These laws are called Kirchhoff's Corollaries of Ohm's Law, and they are expressed by the formulæ

$$\left.\begin{aligned}\Sigma(C) &= 0\\ \Sigma(CR) &= \Sigma(E)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

The difficulty in applying them in practice is to decide beforehand the direction of the current in any conductor.

Hence Maxwell made a great simplification in suggesting that instead of dealing with real currents we should consider imaginary currents all circulating in the same direction, say counter-clockwise, round the meshes, and then take the real

currents in the conductors as the difference of the imaginary currents in adjacent meshes.

This leads to the following rules :—

Assign to each mesh a *mesh symbol*, x, y, z, etc., denoting these imaginary currents.

3. Multiply each mesh symbol by the sum of all the resistances which bound the mesh, and subtract from it the sum of each adjacent mesh symbol, each multiplied by the common resistance, and equate this to the electromotive force or forces acting round the mesh with + or − sign properly added.

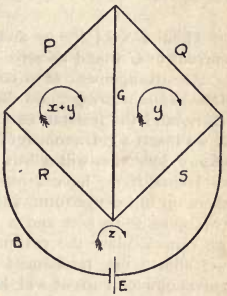


FIG. 16.—Circuit of Wheatstone's Bridge.

For example, consider the case of a 3-mesh network formed by joining 4 points by 6 conductors with 1 E.M.F. in 1 circuit (see Fig. 16). Let P, Q, R, S, G, B be the resistances and E the E.M.F. in conductor B. Then we have 3 meshes bounded by $P+R+G$, $G+Q+S$, and $R+S+B$ respectively. Let $x+y$, y, and z be the mesh symbols, then Maxwell's equations are

$$\left.\begin{aligned}(P+R+G)(x+y) - Gy - Rz &= 0\\ -G(x+y) + (G+Q+S)y - Sz &= 0\\ -R(x+y) - Sy + (R+S+B)z &= E\end{aligned}\right\}, \quad . \quad (2)$$

assuming the E.M.F. is placed only in mesh z.

Hence, rearranging terms we have

$$\left.\begin{aligned}-Rx - (R+S)y + (R+S+B)z &= E\\ -Gx + (Q+S)y - Sz &= 0\\ (P+R+G)x + (P+R)y - Rz &= 0\end{aligned}\right\}. \quad . \quad (3)$$

The real current in the conductor G is $x=(x+y)-y$. Hence if we solve the above equations for x, this gives us the current in G. The above three linear equations can be solved by determinants (see Chap. I.), and the solution is

$$x = \frac{\begin{vmatrix} E, & -(R+S), & (R+S+B)\\ 0, & (Q+S), & -S\\ 0, & (P+R), & -R\end{vmatrix}}{\begin{vmatrix} -R, & -(R+S), & (R+S+B)\\ -G, & (Q+S), & -S\\ (P+R+G), & (P+R), & -R\end{vmatrix}}. \quad (4)$$

If we write out the numerator and denote the denominator by Δ, we have

$$x = E\frac{PS - QR}{\Delta} \quad . \quad . \quad . \quad (5)$$

Now suppose that the resistances P, Q, R, S are in proportion, so that

$$\frac{P}{Q} = \frac{R}{S} \quad . \quad . \quad . \quad (6)$$

Then $PS - QR = 0$, and we should have $x = 0$, or the current in G would be zero.

An arrangement of 6 conductors as in Fig. 16 is called a *Wheatstone's Bridge*, and the conductor G is called the *Bridge Wire*, and the resistances P, Q, R, S the *arms* of the bridge. If we insert a galvanometer in the bridge wire then the absence of any deflection will tell us that the current in G is zero.

Hence if we have 4 resistances P, Q, R, and S, and join them up in lozenge form, and insert a galvanometer G between two opposite corners and a battery B between the other two, we can calculate the current in G by the formula (4), and if we adjust the resistances so that $P : Q = R : S$, then the galvanometer current will be zero, and if 3 of the resistances are known we can calculate the remaining one.

It is usual to have two of these resistances, P and Q, fixed, and in a decimal ratio, *e.g.* 1000 ohms and 1 ohm, or 10,000 ohms and 100 ohms, etc. These resistances P and Q are called the *ratio arms*. The resistance S is then made a variable resistance and can be altered, whilst R is the unknown resistance to be measured. In other cases P and Q are the two sections of one long wire into which it is divided by a slider. The variable resistance S is called the *measuring arm*.

For practical work these arrangements of conductors, called bridges, are made up in various forms.

The Wheatstone's Bridge forms a convenient and much-used method of measuring the ohmic or steady resistance of a wire. The reader will find described in Instrument makers' catalogues a variety of forms of Wheatstone's Bridge.

The most usual form is the so-called Post Office plug pattern of bridge.

In this case the resistances P, Q, S consist of coils of wire wound on bobbins and placed in a box with an ebonite top. On this top are massive brass blocks interconnected by plugs. The coils are connected between the blocks so that on withdrawing a plug the coil is inserted in series.

The ratio arms generally consist of 8 or 10 coils, having each resistances 1, 10, 100, 1000, and perhaps 10,000 ohms, so that by withdrawing 2 plugs we can make $P : Q$ in any decimal ratio from 10,000 : 1 to 1 : 10,000.

The middle and ends of this series of resistances have terminal screws. The measuring arm consists of a series of resistance coils 1, 2, 3, 4, 10, 20, 30, 40, 100, 200, 300, 400, 1000, 2000, 3000, 4000 ohms, so that by withdrawing plugs we can have any resistance up to 11,110 ohms.

This series of coils is connected at one end to the ratio arms, and terminals are provided for joining in the galvanometer, battery, and resistance to be measured.

Also contact keys are generally added in the galvo and battery circuits. To measure a resistance the observer first selects some ratio 10 : 1 or 1 : 10 for the ratio arms $P : Q$, and then withdraws plugs from the measuring arm S until when the battery key is pressed first and afterwards the galvanometer key there is no deflection of the galvo. The resistance R is then given by the equation $\frac{P}{Q} = \frac{R}{S}$ or $R = S\frac{P}{Q}$.

Example.—A resistance of 10 ohms was joined to a Wheatstone Bridge. The ratio arms were 100 : 10, and a plug was taken out of the 5-ohm coil in the measuring arm. An E.M.F. of 2 volts was put on the battery circuit of negligible resistance and a galvo of 10 ohms resistance in the bridge circuit. Calculate the circuit through the galvo.

Answer.—We have $P = 100$, $Q = 10$, $S = 5$, $R = 10$, $B = 0$, $G = 10$ and $E = 2$.

The determinant Δ is therefore

$$\Delta = \begin{vmatrix} -10 & -15 & 15 \\ -10 & 15 & -5 \\ 120 & 110 & -10 \end{vmatrix} = 250 \begin{vmatrix} -1 & -3 & 3 \\ -1 & 3 & -1 \\ 12 & 22 & -2 \end{vmatrix},$$

$$= (6 + 36 - 66 - 108 + 6 - 22)\ 250,$$

$$= -145 \times 250 = -36250.$$

Also $$PS - QR = 100 \times 5 - 10 \times 10 = 400$$

and $$E = 2.$$

Hence $$x = -\frac{2 \times 400}{36250} \text{amps.},$$

$$= \frac{1}{453} \text{ampere (nearly)}$$

or about 2 milliamperes roughly.

To make the galvo current zero or $x=0$ we should have to make $S=1$ and then $PS-QR=100\times 1-10\times 10=0$. The above formula enables us to ascertain the current through the galvanometer or bridge for any value of S, P, Q, and R.

The Wheatstone's Bridge method is suitable for the measurement of resistances of the order of half an ohm up to say 10,000 or 20,000 ohms, but when very low or very high resistances have to be measured or a comparison made between nearly equal resistances, it is not so suitable.

In the case of very small resistances a better method is to employ a standard low resistance, say 0.1 or 0.01 or 0.001 ohm, and to join this in series with the resistance to be measured and send the same current through both.

By means of a potentiometer we can then measure the potential difference of the ends of both the standard and of

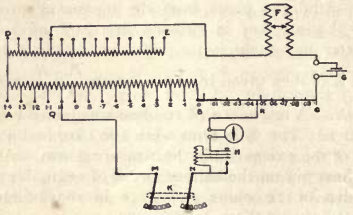


FIG. 17.—Potentiometer Connections.

the unknown resistance and the ratio of these volt drops is the ratio of the resistances.

As a direct reading potentiometer is an instrument of wide utility it will be described here in detail.

On a board or box is stretched a fine wire of manganin or some other material of high resistivity and small temperature coefficient. Underneath the board are fixed 14 coils of wire, each of which has the same resistance as the slide wire, and all coils are joined in series and in series with the slide wire. Also in series with these coils is fixed a variable resistance or rheostat. To the ends of this circuit is applied a steady E.M.F., say from a single secondary cell. By means of the rheostat we can control the current passing through the wire. The junctions of all the 14 coils are brought to a set of contact studs. Also along the wire on the board a sliding contact moves (see Fig. 17).

By means of a radial arm moving over the above-mentioned 14 studs we can connect one terminal of a galvanometer to any junction between two of the 14 coils. The other terminal of the galvanometer, and also a wire brought from the sliding contact, terminate in two working terminals. Figs. 18 and 19 show the appearance of a type of potentiometer made on the above principle by Messrs. Crompton & Co.

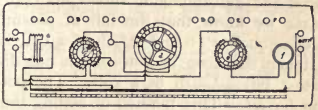


FIG. 18.—Crompton Potentiometer.

In order to use the instrument we require some standard electromotive force, and the one most usually employed is a standard Weston Cadmium Cell giving an E.M.F. of 1.019 volts at 20° C. The E.M.F. diminishes very slightly with rise of temperature at the rate of about 0.004% per degree centigrade, so that the E.M.F. at t°C. can be calculated by the formula

$$E_t = 1.019\{1 - 3.8 \times 10^{-5}(t-20) - 0.065 \times 10^{-5}(t-20)^2\}.$$

This cell consists of an amalgam of cadmium and mercury for one pole and pure mercury for the other. The mercury is covered with mercurous sulphate and the cell is filled up with a saturated solution of cadmium sulphate. For instructions for making it see J. A. Fleming, *Handbook for the Electrical Laboratory and Testing-Room*, vol. i. chap. i.

If the Clark Cell is used in which zinc takes the place of cadmium the E.M.F. is larger, but the temperature coefficient is larger also, so that the E.M.F. at t°C. is

$$E_t = 1.432\{1 - 0.00077(t-15)\}.$$

Having provided a cadmium standard cell, we proceed to calibrate the potentiometer as follows. The slide wire has alongside of it a scale divided into 1000 parts. Assume the temperature to be 20°C. We move the slider to the 190th division on the slide wire. We then set the galvanometer terminal to the junction of the 10th and 11th coil of the 14 coils, so that we have 10 coils and $\frac{190}{1000}$th of one coil in resistance joined in series.

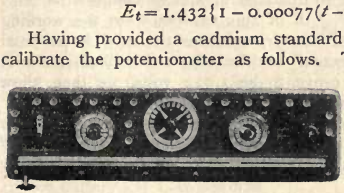


FIG. 19.—Crompton Potentiometer.

If then we join the cadmium cell to the working terminals in such direction that the positive pole of the cell is connected to that end of this series of coils at which the slide wire current enters, and if we vary the current in the slide wire by moving the rheostat, we shall be able to adjust it so that the galvanometer shows no current. When this is the case we know that the fall of potential down the series of coils equal to 10 and $\frac{19}{100}$th of a coil is equal to 1.019 volts. Hence the drop in volts down one division of the slide wire must be $\frac{1}{10000}$th of a volt. The potentiometer is then said to be *set*. This method of setting the potentiometer to make it direct reading was devised by the author in the year 1885.

When the potentiometer is set we can at once read off the value of any other E.M.F. or P.D. which does not exceed 1.5 volts. All we have to do is to connect the points between which we want to know the P.D. to the working terminals of the potentiometer, and then move the slider and contact moving over the 14 studs until the galvo shows no current. Thus, suppose we find that we must insert 2 coils and move the slider to division 842 to obtain no deflection, then this shows that the P.D. is 0.2842 volt.

To measure an electromotive force or potential difference of more than 1.5 volts we have to provide a divided resistance, made as follows: In a box is placed a high resistance, say of 10,000 ohms, which is divided into various sections in the ratio of 9 : 1, 99 : 1, or 999 : 1. The resistance has therefore 2 end terminals and a number of intermediate ones.

Suppose, then, that we desire to measure an E.M.F. of about 100 volts, we connect the extreme terminals of the above-mentioned resistance to this E.M.F., and we connect the short end section, say of $\frac{1}{99}$th of this resistance, to the working terminals of the potentiometer, and measure the fall in potential down this short section in the above-described manner. Suppose it should prove to be 1.012 volts, we then know that the E.M.F. being tested is 100 times this or 101.2 volts.

For full details of the use of the potentiometer the reader is referred to the author's books, *A Handbook for the Electrical Laboratory and Testing-Room*, or to *Electrical Laboratory Notes and Forms*.

By the aid of a set of standard low resistances and a direct reading potentiometer and standard cell, all the measurements of low resistance required in radiotelegraphy can be easily made.

2. Bridge Methods for the Measurement of Small Inductances

The measurement of small inductances used in radio-telegraphic work calls for methods applicable in the case of inductances from a few millihenrys down to a few hundred microhenrys.

One of the best methods of measuring an inductance of a few millihenrys is by the Anderson Bridge as modified by the author.

In this case the inductance to be measured is joined in as one arm in a Wheatstone's Bridge as already described, and the ratio and measuring arms adjusted so as to obtain a balance. The ratio arms of the bridge and also the measuring arm should consist of resistances as free from inductance as possible. In obtaining this resistance balance a galvanometer and a battery may be used in the bridge and E.M.F. circuits as usual.

Suppose the inductance to be measured has a resistance R and an inductance L, then, when balanced on the bridge for steady currents against ratio arms P and Q and measuring arm S, we have as above

$$R = S\frac{P}{Q} \quad . \quad (7)$$

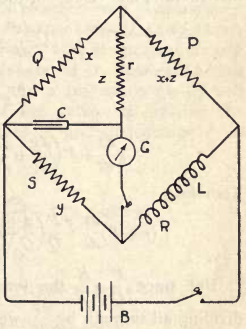


FIG. 20.—The Anderson Inductance Bridge.

When this is done we add a condenser of capacity C which is connected, as shown in Fig. 20, between one corner of the bridge and the galvo terminal, and we add a variable resistance r in series with the galvo. It will then be found that when C and r are removed we can get an adjustment of P, Q, S, and R as in equation (7) so that the galvo does not deflect if the battery key is closed first and the galvo key afterwards.

If, however, we reverse the operation the galvo will give a "kick" or sudden deflection because of the inductance of the arm R. If, however, we add the condenser C and adjust the resistance r to a certain value, it will be found possible to

stop any "kick" or deflection of the galvo whether the battery key or galvo key is closed first. When this is the case there is a certain relation between the four resistances, the inductance of the arm R, and the capacity of the condenser C, and the resistance r in series with the galvo, viz.

$$L = C\{r(R+S)+RQ\}. \quad . \quad . \quad (8)$$

The proof of this formula is as follows: Suppose the galvanometer key to be closed, and that the relation $R = S\frac{P}{Q}$ is satisfied. Then let the battery key be closed. Let x be the quantity of electricity which has flowed through the arm Q, and z the quantity which has passed through the galvo, and $x+z$ that through the arm P, and y that through R from the moment of closing the battery key until the instant considered. Then $\frac{dx}{dt}$, $\frac{dz}{dt}$, $\frac{d(x+z)}{dt}$, $\frac{dy}{dt}$ are the electric currents at the instant considered, and $\frac{z}{C}$ is the potential difference of the terminals of the condenser, provided the adjustment is such that there is no "kick" of the galvo. Hence we have

$$Q\frac{dx}{dt} = \frac{z}{C} + r\frac{dz}{dt},$$

$$\frac{z}{C} = S\frac{dy}{dt},$$

$$r\frac{dz}{dt} + P\left(\frac{dx}{dt}+\frac{dz}{dt}\right) = R\frac{dy}{dt} + L\frac{d^2y}{dt^2}.$$

Therefore

$$(r+P)\frac{dz}{dt} + \frac{P}{Q}\left(\frac{z}{C} + r\frac{dz}{dt}\right) = \frac{Rz}{SC} + \frac{L}{SC}\frac{dz}{dt}$$

But since $\frac{P}{Q} = \frac{R}{S}$ the terms in z cancel out, and hence dividing all through by $\frac{dz}{dt}$, we have

$$r + P + \frac{P}{Q}r = \frac{L}{SC},$$

or

$$L = C\{r(R+S)+RQ\}. \quad . \quad . \quad (9)$$

This method has the great advantage that the alteration in r which is necessary to eliminate the galvo "kick" does

not affect the adjustment necessary to secure the steady balance.

It is necessary for good results that C should bear a certain relation to L. That is, if L is large C must be large, and if L is small C should be small. This adjustment can be found by trial.

If R, S, and Q are measured in ohms, and if C is measured in microfarads, the result will give L in microhenrys.

We can increase the sensitiveness of this method by using, as suggested by the author, a telephone in place of the galvanometer, and a buzzer or interrupter in the battery circuit. We must then use a galvo to obtain the steady balance of the bridge, and after substituting the telephone in the bridge circuit we must adjust the resistance r until no sound is heard in the telephone. In so doing we ought to take notice how much r has to be varied either way to reproduce the least audible sound, because this telephone method tests the acuteness of the observer's hearing as much as it tests the inductance of the coil R.

As an instance of the application of this Anderson method of measuring inductance we give the following.

Example.—A coil of wire was attached to a Wheatstone's Bridge. The ratio arms were $P = 10$ ohms, $Q = 100$ ohms, and the steady balance was found with a measuring arm $S = 1577$ ohms. Then a condenser of 1 microfarad was joined in as shown in Fig. 20, and a resistance $r = 59$ was required to quench the kick of the galvo on pressing the battery key. Find the resistance R and inductance L of this coil.

Answer.

$$R = S \times \frac{P}{Q} = 1577 \times \frac{10}{100} = 157.7 \text{ ohms.}$$

$$L = C\{r(R+S) + RQ\},$$

$$= \frac{1}{10^6}\{59(1577 + 157.7) + 157.7 \times 100\},$$

$$= 0.118 \text{ henry or } 118 \text{ millihenrys.}$$

It should be noted that the resistance measured in the above manner by bridge methods, and also the inductance measured by Anderson's method, is the steady current resistance and inductance, and corresponds to a uniform distribution of the current over the cross-section of the conductor. We have already given in Chapter III. the correction required for

obtaining the high frequency resistance, which is always greater than the steady resistance. On the other hand, the high frequency inductance is slightly less than the steady inductance. We shall in the next chapter give methods for measuring the high frequency inductance. Meanwhile we have given in Chapter III. formulæ for predetermining it in certain cases.

3 Measurement of Mutual Inductance

Another important measurement is that of the mutual inductance of two coils. This may be defined as the number of lines of magnetic force which are linked with both circuits when unit current flows through either of them.

The best method of determining it is by an arrangement due to Professor G. Carey Foster, called a Foster Bridge, although it is not strictly speaking a bridge method. Between four fixed points we join in a condenser of capacity C, a non-inductive resistance P, a battery B, and key K, and galvanometer G, and also the coils R and S, whose mutual inductance M is to be determined (see Fig. 21). Let these two coils have inductances L and N respectively. The condenser C should be one having a variable capacity. We then adjust the capacity C and the resistance P until there is no "kick" or sudden deflection of the galvanometer when the battery key K is depressed.

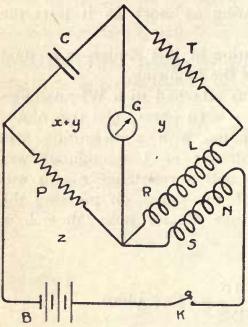


FIG. 21.—Carey Foster Bridge for Mutual Inductance Measurements.

Under these conditions we have

$$M = CRP, \quad . \quad . \quad . \quad (10)$$

where R is the steady resistance of the coil joined in the bridge arm. We can also use the method to determine the capacity of the condenser if we have a pair of coils of which the mutual inductance is known or can be calculated. It should be noted that in joining up the two coils into the bridge

the induced currents must flow in such a direction that the discharge of the condenser opposes that of the coil through the galvanometer when the battery key is raised.

The deduction of the above formula requires an extension of the principles explained in the first section of this chapter.

Maxwell shows in his large *Treatise on Electricity and Magnetism* that if a network of conductors contains condensers and inductances, self or mutual, as well as resistance in its branches, that the network equations must be deduced as follows :—

Let T stand for the *electrokinetic energy* of the system, that is for the sum of such quantities as $\frac{1}{2}Lx^2 + Mxy + \frac{1}{2}Ny^2$, etc., where L and N are inductances of circuits and M their mutual inductance and x and y are the currents in these circuits. Also let F stand for the *dissipation function* which is the sum of such quantities as $Rx^2 + Sy^2$, etc., where R and S are the resistances of the circuits. Then if E is the electromotive force in a circuit Maxwell proved that

$$E = \frac{d}{dt}\left(\frac{dT}{dc}\right) + \frac{1}{2}\frac{dF}{dc}. \quad . \quad . \quad (11)$$

This is analogous to an equation in dynamics called Lagrange's equation, where c stands for any current or *mesh symbol.*

To apply this to the case of the Foster Bridge let $x+y$, y and z be the mesh symbols so that the current through the galvanometer is $x+y-y=x$, and let p stand for $\frac{d}{dt}$, and therefore $\frac{1}{p}$ stands for $\left(\frac{d}{dt}\right)^{-1}$ which last is the inverse operator to $\left(\frac{d}{dt}\right)$ and is hence equal to $\int(\quad)dt$.

Then the kinetic energy T is

$$T = \tfrac{1}{2}Ly^2 - My^2 + \tfrac{1}{2}Nz^2, \quad . \quad . \quad (12)$$

since the current in the resistance R which has an inductance L is y, and the current in the battery circuit which has an inductance N is z, and the mutual inductance is M. Also the dissipation function is

$$F = P(x+y-z)^2 + G(x+y-y)^2 + Ry^2 + Sz^2, \quad (13)$$

assuming the battery cell has no resistance.

Then differentiating the equations (12) and (13) with respect

to each mesh symbol, and remember that the charged condenser has a back voltage or back E.M.F. which is equal to

$$\frac{1}{C}\int(x+y)\,dt = -\frac{x+y}{Cp},$$

we have the cyclical equations from (11)

$$\left.\begin{array}{ll} \text{for the } z \text{ mesh} & Npz - Mpy + Pz - P\overline{x+y} + Sz = E \\ \text{for the } y \text{ mesh} & Lpy - Mpz + Gy - G\overline{x+y} + Ry = 0 \\ \text{for the } \overline{x+y} \text{ mesh} & P\overline{x+y} - Pz + G\overline{x+y} - Gy = -\dfrac{x+y}{Cp} \end{array}\right\}. \quad (14)$$

Rearranging in x, y, and z and solving (14) by determinants we obtain the solution

$$x = \frac{E\begin{vmatrix} R+Lp, & -Mp \\ P+\dfrac{1}{Cp}, & -P \end{vmatrix}}{\Delta} \quad . \quad . \quad (15)$$

where Δ is the determinant of the above three equations (14).

Now in order that x the galvanometer current may be zero we must have the numerator in (15) zero or

$$\left(\frac{M}{C}R - P\right) + Pp(M-L) \quad . \quad . \quad (16)$$

must be zero.

Hence if the E.M.F. E is not a function of the time, it is sufficient that $M = CRP$ for the galvo current to be zero. This is the case when the electromotive force is provided by a battery cell. If, however, the E.M.F. is an alternating current, then for zero galvanometer current it is necessary that we should have $M = L$ as well as $M = CRP$. Otherwise we can insert a resistance in series with the condenser.

As an illustration of the use of the above formula we take the following.

Example.—A certain induction coil had a secondary or fine wire circuit having a resistance of 7394 ohms, and a primary or thick wire circuit of resistance 0.58 ohm. It was joined into a Foster Bridge with a condenser having a capacity of 4.926 microfarads. It was found that a resistance of 32 ohms was required in the adjacent bridge arm to annul the "kick" of the galvanometer on breaking and making the battery circuit. Find the mutual inductance of the primary and secondary coils.

Answer.—We have $R = 7394$ ohms, $P = 32$ ohms, and $C = \frac{4.926}{10^6}$ farads.

Hence

$$M = \frac{7394 \times 32 \times 4.926}{10^6} = 1.1896 \text{ henrys.}$$

This large mutual inductance is due to the large number of turns in the secondary circuit.

This method may be also used for the determination of capacity provided we have a variable standard of mutual inductance such as that made by A. Campbell.

4. Coefficients of Coupling

When two circuits are placed near each other so that they have a mutual inductance M, and if each of them has self-inductance L and N, then the ratio of the mutual inductance to the geometric mean of the two self-inductances is called the *Coefficient of Coupling*. It is denoted by k, so that

$$k = \frac{M}{\sqrt{LN}} \quad . \quad . \quad . \quad . \quad (17)$$

It is expressed as a percentage so that a coupling of 10 per cent means that M is $\frac{1}{10}$th of $\sqrt{LN}$ in magnitude or arithmetic value, when L, M, and N are measured in the same units. Hence to measure k we require to measure L, M, and N for the two coils. It should be borne in mind that L signifies the number of lines of magnetic flux or force which are self-linked with the primary coil when unit current flows through it. Also N signifies the same for the secondary coil. Again M signifies the number of lines of force due to unit current in the primary which are linked with both primary and secondary. Hence the total linkage of the circuit and the flux is either

$$\begin{matrix} L + 2M + N \\ \text{or} \quad L - 2M + N, \end{matrix} \quad . \quad . \quad . \quad (18)$$

according to which way the unit current flows in the windings of the two coils. If, then, we measure by the Anderson method the effective inductance of the two coils joined in series, first with the currents circulating the same way in each coil and next with the currents circulating in opposite directions, and call these effective inductances L_1 and L_2, we have

$$L_1 = L + 2M + N \quad . \quad . \quad (19)$$

and

$$L_2 = L - 2M + N. \quad . \quad . \quad (20)$$

Therefore $$\frac{L_1 - L_2}{4} = M \quad . \quad . \quad . \quad (21)$$

and $$\frac{L_1 + L_2}{2} = L + N. \quad . \quad . \quad . \quad (22)$$

If therefore we measure L and N separately, we have a check on the value of $L_1 + L_2$, and we can infer the coefficient of coupling k from the measurements made since it is

$$k = \frac{L_1 - L_2}{4\sqrt{LN}} \quad . \quad . \quad . \quad (23)$$

Example.—Two equal square coils, each coil consisting of eight turns wound on a frame of side equal to 64.5 cms., were placed near to and parallel to each other and the following measurements made by the Anderson-Fleming method with bridge, buzzer, and telephone.

The inductance of the two coils in series joined up with currents circulating in the same direction was 287,800 cms. With currents circulating in opposite directions it was 180,700 cms.

The sum of the inductances with coils far apart was found to be 234,600 cms., and that of each coil alone was 116,200 cms. Find the mutual inductance in the first position and the coefficient of coupling.

Answer.—We have by measurement

$$L = 116{,}200 \text{ and } N = 116{,}200$$
$$L + N = 234{,}600,$$

also
$$L + 2M + N = 287{,}800$$
$$L - 2M + N = 180{,}700.$$

Hence $$M = \frac{107{,}100}{4} = 26{,}775 \text{ cms.}$$

and $L + N = 234{,}200$, which agrees fairly well with the measured value.

Therefore the coefficient of coupling k is

$$k = \frac{26{,}775}{\sqrt{(116{,}200)^2}} = 0.23 = 23\%.$$

A coupling of 23 per cent is called *close coupling*. In fact anything over 10 per cent is close, and under 10 per cent is *weak coupling*

5. General Conditions for Equilibrium in Bridge Measurements

If we suppose the arms of a Wheatstone's bridge to be filled in with inductive resistances, or with condensers having capacity in series with non-inductive or inductive resistances, or, generally speaking, with any arrangement of coils, condensers, or resistances, and if we suppose an alternating E.M.F. of simple sine wave form is applied between two diagonal points on the bridge, and if a telephone is connected between the other diagonal points, we can state the condition for zero bridge current or silence in the telephone as follows:

The vector impedances of the four arms must be in proportion. Let us call the four arms of the bridge P, Q, R, S, and let there be in arm P a condenser of capacity C, in series with a non-inductive resistance r.

In the arm Q let there be a coil having an inductance L, and a resistance R. In the branches R and S, let there be leaky condensers of capacity C_1 and C_2 with leakances S_1 and S_2. If, then, we apply a simple periodic E.M.F. to one pair of diagonals, the condition for zero current in the conjugate diagonal or bridge current is as follows:

The vector impedance of the arm P is $r-\frac{j}{Cp}$, and that of arm Q is $R+jpL$, and that of branches R and S is

$$(S_1+jpC_1)^{-1} \text{ and } (S_2+jpC_2)^{-1},$$

where $p=2\pi$ times the frequency of the E.M.F.

Hence the condition for equilibrium is

$$\frac{r-\frac{j}{Cp}}{R+jpL}=\frac{S_2+jpC_2}{S_1+jpC_1}, \quad . \quad . \quad (24)$$

or

$$\left(r-\frac{j}{Cp}\right)(S_1+jpC_1)=(R+jpL)(S_2+jpC_2). \quad (25)$$

If, then, we multiply out both sides of this last equation (25), we shall have on each side a complex quantity of the form $A+jB$, and the equation will take the form

$$A+jB=A^1+jB^1.$$

From which we can infer that

$$A = A^1 \text{ and } B = B^1.$$

Hence there are two conditions to be fulfilled in order that the bridge current may be zero, or a telephone in that bridge circuit give no sound when the bridge is supplied with alternating currents of simple sine form.

M. Wien has given many forms of bridge connection which by the above method can be used for determining the conductivity and capacity of leaky condensers, or condensers made with imperfect dielectrics.

CHAPTER VII

HIGH FREQUENCY CYMOMETER MEASUREMENTS, WAVE-LENGTH AND DECREMENT

1. Oscillatory Circuits; Resonance.

A CIRCUIT which consists of a condenser of capacity C in series with a coil of wire of low or negligible resistance R and inductance L is called an oscillatory circuit.

If a charge is given to the condenser and then left to discharge, or if a sudden impulsive electromotive force is applied to the circuit, electricity oscillates in it with a frequency n such that

$$n = \frac{1}{2\pi}\sqrt{\frac{1}{CL} - \frac{R^2}{4L^2}} \qquad (1)$$

We have already proved this formula in Chapter IV., see equation (50).

If R is numerically small compared with $2L$, so that $\frac{R^2}{4L^2}$ can be neglected in comparison with $\frac{1}{CL}$, then the expression (1) reduces to

$$n = \frac{1}{2\pi\sqrt{CL}}. \qquad . \quad . \quad . \quad (2)$$

In this expression the quantity $\sqrt{CL}$ is called the *oscillation constant* and denoted by O. Hence

$$n = \frac{1}{2\pi O} = \frac{1}{6.283O}. \qquad . \quad . \quad (3)$$

Therefore the time period T of one complete oscillation is $\frac{1}{n}$ and

$$T = 6.283O. \qquad . \quad . \quad . \quad (4)$$

Circuits which have the same oscillation constant have the same time period of oscillation and are said to be *equiperiodic*, or *isochronous*, or *resonant*, or *syntonic* circuits, or, in simpler language, are said to be *in tune* with each other.

Since the inductances used in radio-telegraphy are generally of such order as to be measured in absolute electromagnetic units or centimetres, and since the capacities are of such order as to be measured in microfarads we note that if C_{mfds} is capacity measured in mfds. and L_{cms} is inductance measured in centimetres we have

$$n = \frac{1}{2\pi\sqrt{\frac{C_{mfds}}{10^6} \cdot \frac{L_{cms}}{10^9}}},$$

or

$$n = \frac{\sqrt{1000}}{2\pi} \cdot \frac{10^6}{\sqrt{C_{mfds} \cdot L_{cms}}},$$

or

$$n = \frac{5.033 \times 10^6}{\sqrt{C_{mfds} \cdot L_{cms}}}. \quad . \quad . \quad . \quad (5)$$

Example.—A charged Leyden jar whose capacity is $\frac{1}{400}$ mfd. is discharged through a wire bent in the form of a square which has an inductance of 1600 cms. Find the time period of one complete oscillation.

Answer.—We have $C_{mfds} = \frac{1}{400}$ $L_{cms} = 1600$, hence

$$n = \frac{5 \times 10^6}{\sqrt{\frac{1}{400} \cdot 1600}} = 2.5 \times 10^6,$$

or $T = 0.4$ of one-millionth of a second.

It is convenient to call one-millionth of a second a *microsecond*. Hence in this case the time period of oscillation is $\frac{2}{5}$ths of a microsecond.

A microsecond is as much smaller than a second as a second is than a fortnight.

If oscillations are set up in an oscillatory circuit, and if there is any source of energy dissipation in it due to resistance or other cause, the oscillations will die away or be damped. The Napierian logarithm of the ratio of the amplitude of one oscillation to the next in the opposite direction is called the

logarithmic decrement or *decrement* (δ) of the circuit per semi-period. Hence

$$\delta = log_e \frac{I_1}{I_2} = 2.3026 \, log_{10} \frac{I_1}{I_2} \quad . \quad . \quad . \quad (6)$$

The decrement per complete period is double that per semi-period.

If we consider the oscillations to be practically extinguished when the amplitude has decreased to 1 per cent of the initial one, then if m is the number of semi-oscillations in a train

$$m = \frac{4.6052 + \delta}{\delta}, \quad . \quad . \quad . \quad (7)$$

where δ is the decrement per semi-period.

Hence if the decrement per semi-period is 0.1 or per complete period 0.2, there will be $\frac{4.6 + 0.1}{0.1} = 46$ semi-oscillations or 23 complete oscillations in a train or before the amplitude has fallen to 1 per cent of the initial.

If the only source of dissipation of energy in a circuit is the high frequency resistance R' of the inductance coil, and if L' is the inductance for the frequency n, then the decrement is given by

$$\delta = \frac{R'}{4n L'} \quad . \quad . \quad . \quad (8)$$

Every oscillatory circuit is therefore distinguished by a certain *oscillation constant* $O = \sqrt{\frac{C}{L}}$ and a certain *decrement*.

We have already shown that a simple harmonic oscillation can be mathematically represented by the real part or horizontal step of the quantity ϵ^{jpt} where $p = 2\pi n$ and n is the frequency.

If p is a real quantity, this is equal to $cos\, pt + j\, sin\, pt$.

If, however, p is the real part of a complex quantity P such that $P = p + j\alpha$, then the real part of ϵ^{jpt} represents mathematically a damped oscillation. For $P = p + j\alpha$, hence $\epsilon^{jpt} = \epsilon^{-\alpha t}\epsilon^{jpt}$, which is equal to $\epsilon^{-\alpha t}(cos\, pt + j\, sin\, pt)$, and the real part of this last is the function $\epsilon^{-\alpha t}\, cos\, pt$, which represents a damped oscillation. The decrement δ of this oscillation per semi-period is equal to $\alpha \frac{\tau}{2}$ where τ is the periodic time, and hence

$$\alpha = 2n\delta.$$

If the condenser used in the oscillation is a leaky condenser having a conductivity or leakance S, or if it has a perfect

dielectric but has a shunt resistance r across its terminals such that $r=\frac{1}{S}$, then we can obtain the time period of oscillation and the decrement as follows: If the inductance coil in series with the condenser has a resistance R and an inductance L, and if the condenser has a capacity C, then when the oscillations are established the drop in voltage down the inductive resistance expressed as a complex quantity is $(R+jPL)I$. Also the drop in voltage down the leaky condenser is $(S+jPC)^{-1}I$ where I is the current. Hence, since there is no impressed E.M.F. when the oscillations are free, we have the equation

$$(R+jPL)+\frac{1}{S+jPC}=0,$$

or

$$(R+jPL)(S+jPC)=-1=j^2.$$

Multiply out the left hand side, divide all through by CL, and put $2a$ for R/L and $2b$ for S/C and we have

$$(jP+2a)(jP+2b)=\frac{j^2}{CL}.$$

Add $(a-b)^2$ to both sides and rearrange and we have

$$\left\{jP+(a+b)\right\}^2=j^2\left\{\frac{1}{CL}-(a-b)^2\right\}.$$

Hence, taking the square root, we have

$$P=p+ja=j(a+b)\pm\sqrt{\frac{1}{CL}-(a-b)^2},$$

or, replacing the values of a and b and equating real and unreal parts, we have

$$\alpha=(a+b)=2n\delta$$

$$p=2\pi n=\sqrt{\frac{1}{CL}-\left(\frac{R}{2L}-\frac{S}{2C}\right)^2}.$$

Accordingly the decrement and frequency for a condenser circuit containing a leaky or shunted condenser are given by

$$\delta=\frac{1}{2n}\left(\frac{R}{2L}+\frac{S}{2C}\right). \quad . \quad . \quad (9a)$$

$$\text{and } n=\frac{1}{2\pi}\sqrt{\frac{1}{CL}-\left(\frac{R}{2L}-\frac{S}{2C}\right)^2} \quad . \quad . \quad (9b)$$

It will be seen, therefore, that conductivity in the condenser affects both the decrement and the frequency.

If two oscillation circuits are put near each other, and if their coils or circuits have a mutual inductance, and if oscillations are set up in one circuit, then oscillations will be induced in the other circuit. If the circuits have different oscillation constants or are out of tune the secondary oscillations induced may be very feeble. If the constant of the secondary circuit is gradually altered so as to tune it to the other, then the current in it will rise and become a maximum when the two circuits are in tune. This can be done by altering the capacity or inductance of the secondary circuit.

If we take a measurement of the current in the secondary circuit by a hot wire ammeter and note also the value of the oscillation constant at that moment, we can plot a curve in which the ordinates represent the current and the abscissa the corresponding natural frequency of the secondary circuit or the value of $\frac{1}{2\pi\sqrt{C_2 L_2}}$ for the secondary circuit. This curve is called a *resonance curve*. It rises to a maximum value and then falls again, and the maximum value occurs for that value of the constant of the secondary circuit for which it is in tune with the primary. The resonance curve has a sharp peak when the two circuits have small decrements, and a rounded summit when they have large decrements (see Fig. 22). An oscillation circuit in which the capacity and inductance or both can be altered so as to give the circuit any known and required oscillation constant is called a *cymometer* or often a *wave meter*.

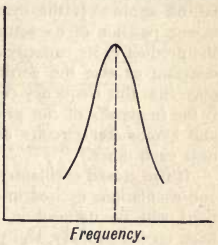


FIG. 22.—A Resonance Curve.

2. Frequency Measurement

A cymometer can be used to determine the frequency of the oscillations in another oscillatory circuit as follows:

Place the cymometer near to the circuit in which the oscillations are taking place so that secondary oscillations are generated in it. The cymometer must then be provided with some means of indicating when the current in its own circuit is a maximum. This can be done by inserting in its circuit a short length of fine wire which is heated by the oscillations.

Against this wire is pressed a thermoelectric junction of bismuth and iron or iron and constantan, and this thermo-junction is connected to a low resistance galvanometer, or else a vacuum tube, filled preferably with rarified Neon, has its terminals connected to the plates of the condenser in the cymometer circuit, as in the author's cymometer (see Frontis-piece).

If then the capacity or inductance in the cymometer circuit is gradually altered, it will be found that the current in it, as indicated by the thermal ammeter, or else the P.D. at the terminals of its condenser, as indicated by the glow of the Neon tube, will gradually increase to a maximum and then fall off again. If the cymometer is calibrated so as to show in any position of its setting the value of the square root of the product of its capacity and inductance, that is its oscillation constant or else the corresponding frequency, we can say at once that this frequency corresponding to its maximum current is the frequency of the primary current provided the primary and cymometer circuits are feebly coupled, that is not too near each other.

If two closed oscillatory circuits are placed in close coupling and oscillations excited in one of them, then secondary oscilla-tions will be induced in the other. These again will act back on the primary, and the two circuits will act and react on each other, and will set up in one another a complex oscillation in which the amplitude does not remain constant or decay steadily, but rises and falls periodically in such fashion that when the oscillations are at their maximum amplitude in one circuit they are at zero or a minimum in the other and *vice versa*. These amplitudes vary as shown in the curves in Fig. 23.

A periodic curve, of which the amplitude waxes and wanes as shown in the bottom curve of Fig. 23, can be resolved into the sum of two simple periodic curves of different frequencies.

For if $y_1 = Y_1 \sin pt$ represents one periodic curve of frequency n_1 such that $2\pi n_1 = p$, and if $y_2 = Y_2 \sin qt$ represents another periodic curve of frequency n_2 such that $2\pi n_2 = q$, and if p is not very different from q so that $\frac{p+q}{2}$ can be considered nearly equal to p, and also if $Y_1 = Y_2 = Y$, or the amplitudes are nearly the same, then we have by trigonometry,

$$Y \sin pt + Y \cos qt = \left[2Y \cos \frac{p-q}{2} t \right] \sin pt.$$

In other words, the sum of two simple harmonic motions of slightly different frequency is also a simple harmonic motion of the mean frequency, but with an amplitude which increases and diminishes with a frequency which is equal to the difference of the frequencies of the two separate motions.

Thus if we superimpose on an alternating current of amplitude 10 having a frequency of 100, another current of nearly the same amplitude, but frequency 95, then we obtain an alternating current of a frequency of 97.5 of which the amplitude varies from zero to 20, 5 times in a second.

The effect is analogous to the phenomenon of *beats* in music when two organ pipes slightly out of tune or differing in frequency are sounded together. The resultant sound waxes and wanes in magnitude.

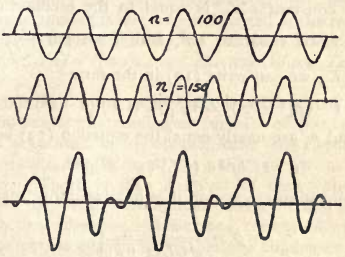


FIG. 23.—Resultant of Two Superimposed Oscillations showing Beats.

The electrical equivalent of this effect is the increase and decrease in current which takes place in the trains of oscillations, whether persistent or damped, when two oscillating circuits act inductively on each other.

The frequencies of the two electrical oscillations into which the complex oscillation can be resolved may be determined as follows :

Let one circuit have a resistance R_1, an inductance L_1, and a condenser of capacity C_1 in series with it, and let the same quantities for the secondary circuit be R_2, L_2, and C_2. Let $p = 2\pi n$, where n is frequency and for the two circuits let p_1 and p_2 denote $2\pi n_1$ and $2\pi n_2$. Let M be the mutual inductance and I_1 and I_2 the two currents.

Then if the oscillations are very feebly damped or undamped we have the equations,

$$\left\{R_1 + jp_1L_1 + \frac{1}{jp_1C_1}\right\} I_1 + jMp_2I_2 = 0. \quad . \quad (10)$$

$$\left\{R_2 + jp_2L_2 + \frac{1}{jp_2C_2}\right\} I_2 + jMp_1I_1 = 0. \quad . \quad (11)$$

These equations merely express the fact that there is no impressed E.M.F. in either circuit when the oscillations are once started.

If we eliminate I_1 and I_2 from the above equations, and if we assume the resistances R_1 and R_2 to be small in comparison with the reactances p_1L_1 and p_2L_2 we obtain the equation

$$M^2C_1C_2p_1^2p_2^2 = C_1C_2L_1L_2p_1^2p_2^2 + 1 - p_1^2C_1L_1 - p_2^2C_2L_2. \quad (12)$$

If then we consider the circuits to be tuned so that the oscillation constant C_1L_1 is equal to the oscillation constant C_2L_2 and both equal to CL, and if we put $\frac{M}{\sqrt{L_1L_2}} = k$ or $M^2 = k^2L_1L_2$, we can write (12) in the form

$$1 - (p_1^2 + p_2^2)CL + p_1^2p_2^2(CL)^2(1 - k^2) = 0. \quad (13)$$

If p_1 and p_2 are nearly equal the equation (13) becomes

$$1 - 2CLp^2 + (CL)^2(1 - k^2)p^4 = 0,$$

and this biquadratic equation in p has two roots, viz.

$$\left.\begin{aligned} p' &= \frac{1}{\sqrt{CL(1+k)}} \\ p'' &= \frac{1}{\sqrt{CL(1-k)}} \end{aligned}\right\} \quad . \quad . \quad . \quad (14)$$

This shows that there are set up in both circuits oscillations of two frequencies, one greater and one less than the natural frequency n of each circuit alone. For we have from equation (2)

$$n = \frac{1}{2\pi\sqrt{CL}}, \quad . \quad . \quad . \quad (15)$$

and from (14)

$$p' = 2\pi n' = \frac{1}{\sqrt{CL}\sqrt{1+k}}, \quad . \quad . \quad (16)$$

$$p'' = 2\pi n'' = \frac{1}{\sqrt{CL}\sqrt{1-k}}. \quad . \quad . \quad (17)$$

Hence from (15), (16), (17) we have

$$\left.\begin{aligned} n' &= \frac{n}{\sqrt{1+k}} \\ n'' &= \frac{n}{\sqrt{1-k}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (18)$$

Accordingly if the coupling is very loose k is very small, and then $n' = n'' = n$, or there is only an oscillation of one single frequency; but if k is, say, 0.5 then n' and n'' are very different in value.

The moral of this is that in using a cymometer to determine the frequency in an oscillation circuit the coupling of the cymometer and of the oscillation circuit must be very feeble or not more than 5 per cent.

The above theory can be extended to damped oscillations in which the decrement is not extremely small or the resistances negligible, but for the full theory the reader is referred to the author's book, *The Principles of Electric Wave Telegraphy and Telephony.*

Various forms of wave meter have been invented. In the Fleming cymometer the condenser consists of two concentric brass tubes separated by an ebonite tube (see Frontispiece). The outer brass tube slides off the inner one so as to vary the capacity. There is also a spiral of bare copper wire in one layer of nearly contiguous turns. This is placed parallel with the tubular condenser and connected with it by a sliding contact, in such manner that the condenser capacity varies proportionately with the length of the spiral included in circuit with it when the outer brass tube is slid off the inner. The circuit is completed by a copper bar. Across the terminals of the condenser is placed a Neon vacuum tube which glows more or less brightly when oscillations are set up in the circuit. Or else a thermoelectric ammeter can be inserted in series with the condenser. We have, therefore, an oscillation circuit, consisting of a variable capacity C and a variable inductance L and means for detecting the mean square value J of the current in this circuit. The instrument has a scale by which we can read off directly the value of the square root of the product C (in microfarads) and L (in centimetres) and also the value of the natural frequency

$$n_2 = \frac{5 \times 10^6}{\sqrt{C_{\text{mfds.}} \times L_{\text{cms}}}} \quad . \quad . \quad . \quad (19)$$

for any setting of the cymometer.

In another form of wave meter made by Marconi's Wireless Telegraph Company the variable capacity is a multiple plate

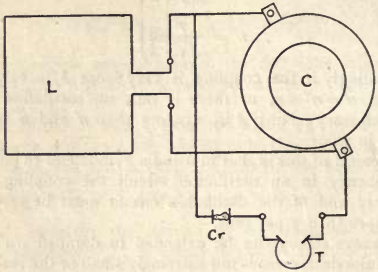


FIG. 24.—The Circuits of the Marconi Wavemeter.

semicircular disk variable condenser and the inductance is a fixed rectangular coil in the lid of a box containing the condenser. The detector is a carborundum crystal in series with a high resistance telephone which is connected as a shunt across the terminals of the variable condenser (see Figs. 24 and 25).

The Marconi wavemeter or the Fleming cymometer can be used for the following measurements.

(1) *To measure the frequency of the oscillations in any circuit within the range of the instrument.*

Place the cymometer so that its copper bar is near to and parallel with the circuit or wire in which exist the oscillations of which we desire to know the frequency. Join in the Neon tube to the holder connected to the terminals of the condenser. Slide the condenser tube along until the Neon tube glows most brightly. Then move the cymometer bar away from the circuit under test until it is found that the Neon tube only

FIG. 25.—The Marconi Wavemeter.

just glows when the cymometer condenser is adjusted to one particular position. Then read off on the scale the corresponding oscillation constant $O = \sqrt{CL}$ and the corresponding frequency $n = \frac{5 \times 10^6}{\sqrt{CL}} = \frac{5 \times 10^6}{O}$.

If there are oscillations of two frequencies present in the circuit then the Neon tube will glow brightly when the cymometer condenser is set to two positions, the oscillation constants of which correspond to these frequencies.

(2) *To measure the wave length of the electric waves sent out from any radiator or aerial wire.*

Place the cymometer with its bar parallel and near to part of the radiating circuit or aerial wire and move along the condenser until the Neon tube glows most brightly, using the same precautions as in the frequency test. Then observe the oscillation constant for that setting.

In the case of all wave motions the wave velocity W is connected with the wave length λ and the wave frequency n as follows:

$$W = n\lambda. \quad . \quad . \quad . \quad (20)$$

In the case of electric waves in space the wave velocity is 3×10^{10} centimetres per second or 300,000 kilometres per second or nearly 1000 million feet per second.

Hence we obtain the wave length in metres by dividing 3×10^8 by the wave or oscillation frequency n. Now the oscillation frequency n is connected with the oscillation constant $O = \sqrt{CL}$ by the relation

$$n = \frac{5 \times 10^6}{O}. \quad . \quad . \quad . \quad (21)$$

Hence from (20) and (21)

$$\lambda = \frac{3 \times 10^8}{5 \times 10^6} O,$$

or $$\lambda = (\text{wave-length in metres}) = 60\ O, \quad . \quad (22)$$

where O is the oscillation constant of the cymometer when in tune with the circuit under test.

If we work in feet then

$$\lambda = (\text{wave-length in feet}) = 200\ O. \quad . \quad (23)$$

These formulæ are only approximate, when great exactness is required we must put for (22) and (23) the expressions

$$\left.\begin{aligned}\lambda\,(\text{metres}) &= 59.6\ O\\ \lambda\,(\text{feet}) &= 195.56\ O\end{aligned}\right\} \quad . \quad . \quad (24)$$

where O is the oscillation constant. Also from (18). If there are waves of two wave-lengths λ_1 and λ_2 emitted from a tuned coupled transmitter, and if λ is the natural wave-length of the antenna, then we have

$$2\lambda = \lambda_1 + \lambda_2. \quad . \quad . \quad . \quad (24a)$$

(3) *To employ the cymometer to measure an inductance or a capacity.*

We can employ a wavemeter or cymometer which has been carefully standardised as a means of measuring a capacity or an inductance for high frequency as follows:

An oscillatory is set up which consists of a condenser comprising one or more Leyden jars. These are connected in series with a rectangular circuit formed by laying insulated copper wire, say of No. 14 S.W.G. size, round the edge of a rectangular board. This wire should be interrupted in two places; in one a spark gap should be inserted, and in the other gap the condenser should be inserted, so that condenser, spark gap, and rectangular circuit are in series with each other. It is convenient also to have another gap in the rectangular wire which can be bridged over or in which can be inserted any loop of wire of which the inductance is required. The spark gap in the above circuit is connected to an induction coil so as to set up damped oscillations in the rectangular circuit.

The cymometer is placed near to this rectangular circuit with its copper bar parallel to one long side of the rectangle and adjusted to loose coupling. It will then be found that on taking a cymometer reading oscillations of one single frequency exist in the rectangle, and these have a frequency n, and the circuit has an oscillation constant $O = \sqrt{CL}$ of a certain numerical value as read on the scale of the cymometer when the latter is adjusted, so that the Neon tube glows most brightly.

This means that the product of the inductance L of the rectangular circuit reckoned in centimetres, and the capacity reckoned in microfarads, has a certain numerical value equal to O^2. Now suppose the capacity in the rectangular circuit is increased by the addition of another small unknown capacity

C_1, which is joined in parallel with the Leyden jars of capacity C, then if we take another reading of the cymometer we observe a fresh oscillation constant O_1. Hence we have

$$O^2 = CL,$$
$$O_1^2 = (C + L_1)L,$$
$$= O^2 + C_1 L,$$

or

$$C_1 = \frac{O_1^2 - O^2}{L}. \quad . \quad . \quad . \quad (25)$$

Now the inductance L of the rectangular circuit can be calculated from the dimensions of the rectangle and the thickness of the wire by the formula (44) given in Chapter III. Hence when we know the value of this inductance L we can calculate by (25) the value of C_1.

In the Fleming cymometer the lid of the box which contains it has a rectangular wire circuit with two gaps in it extended into short connecting wires attached to it. The calculated value in centimetres of this rectangular inductance is given with the instrument. Suppose it to be 6000 centimetres, and let the two readings O and O_1 be respectively 3 and 5, then we should have

$$C_1 = \frac{25 - 9}{6000} = 0.002666 \text{ mfd.},$$

or 2666 micromicrofarads.

3. Delineation of Resonance Curves, Measurement of Decrement

The cymometer or wavemeter enables us to delineate a resonance curve for any oscillatory circuit, and also to determine the logarithmic decrement of the oscillations.

To delineate a resonance curve the cymometer or the wavemeter must have in series with its circuit some form of sensitive hot wire ammeter which enables us to determine the R.M.S. value of the current in that circuit. A convenient form is that devised by the author for use with the cymometer.

Between two metal pins a short length (about 1 or 1.5 cms.) of very fine constantan or high resistance wire is stretched. To the centre of this is soldered a sensitive thermo-electric junction of bismuth and iron. Some skill is necessary in making this junction, as the contact between the iron and bismuth must just meet on the fine constantan wire. The bismuth and iron wires are prolonged and connected to two

terminals, and these are connected to a low resistance galvanometer. The first step is to calibrate this thermo-ammeter by passing through it known continuous currents and observing the corresponding galvanometer deflections. We can then construct a calibration curve in which the horizontal abscissæ are the true currents and the vertical ordinates the corresponding scale deflections of the galvanometer. This curve will be nearly a parabola, because the scale deflections are approximately proportional to the square of the currents. Hence we can ascertain from this curve, and from the galvanometer deflections, the ampere-value or R.M.S. value of any train of electric oscillations. Let this value be denoted by J.

If then we have damped electric oscillations set up in any circuit comprising a spark gap, condenser, and inductance, and if we place a cymometer or wavemeter near to it this wavemeter circuit being provided with a thermal ammeter, we can read off the R.M.S. value of the secondary current induced in the cymometer circuit. In making this measurement the cymometer must be approached cautiously to the circuit under test so as not to run the risk of overheating and fusing the fine wire in the ammeter. If then we begin with the cymometer adjusted so as to be very much out of tune with the primary circuit the current J in the cymometer will be very small. If we bring the cymometer gradually into tune that current will increase to a maximum denoted by J_{max}. We can observe at every stage the value of the oscillation constant O, and therefore the frequency n_2 for that setting of the cymometer which corresponds to a current J in it, for,

$$n_2 = \frac{5 \times 10^6}{O}. \quad . \quad . \quad . \quad (26)$$

If then we bring the current gradually to a maximum we have a certain oscillation constant O_1 which corresponds to a frequency n_1, for which the cymometer current is J_{max}. We can then calculate the value of $1 - \frac{n_2}{n_1} = x$ and the value of $\frac{y}{\sqrt{1-y^2}}$, where $y = \frac{J}{J_{max}}$.

Also we can plot a curve the abscissæ of which are $\frac{n_2}{n_1}$ and the ordinates the value of J corresponding to n_2. We then have a curve which rises up to a peak and falls down again (see Fig. 26). This curve is called a resonance curve.

If then d_1 is the logarithmic decrement per semiperiod of the primary circuit, and d_2 that of the cymometer circuit a formula given by Bjerknes for the sum of $d_1 + d_2$ is as follows:—

$$X = d_1 + d_2 = \pi\left(1 - \frac{n_2}{n_1}\right)\frac{J}{\sqrt{J_{max}^2 - J^2}}, \quad . \quad (27)$$

or

$$X = 3.1415\, x\frac{y}{\sqrt{1-y^2}}. \quad . \quad . \quad (28)$$

The proof of this formula is long, and for it the reader must be referred to the author's book, *The Principles of Electric Wave Telegraphy and Telephony.*

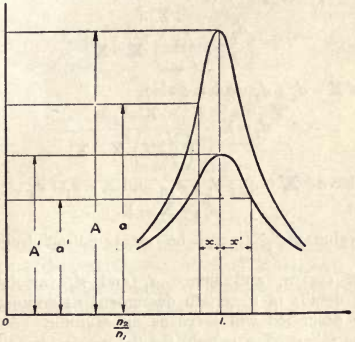


FIG. 26.—Resonance Curves employed for Determination of the Decrement.

The formula is only accurate and valid provided that we do not take values of $\frac{n_2}{n_1}$ which exceed 1.05 or fall below 0.95. In other words, it only applies just at the peak of the resonance curve.

In order to separate out that part of the total decrement due to the cymometer itself we have to make an additional experiment with the cymometer circuit resistance increased by a certain known amount. This is done by introducing into the cymometer circuit a short piece of fine high-resistance wire. We then take a second resonance curve, which falls inside the first one (see Fig. 26).

The addition of this resistance increases the total decrement

by an amount d'_2, and we can, from the second resonance, find in the same manner the total decrement

$$\mathbf{X}' = d_1 + d_2 + d'_2 = 3.1415\, x' \frac{y'}{\sqrt{1 - y'^2}}, \quad . \quad (29)$$

where x' and y' are taken from the second resonance curve.

Now let J_{max} and J'_{max} be the maximum values of the currents, or the resonance currents as they are called, in the two cases, then Bjerknes and Drude have shown that

$$J^2_{max}(d_1 + d_2)d_2 = J'^2_{max}(d_1 + d_2 + d'_2)(d_2 + d'_2), \quad (30)$$

or

$$J^2_{max} d_2 \mathbf{X} = J'^2_{max}(d_2 + d'_2)\mathbf{X}'. \quad . \quad . \quad (31)$$

Hence from (30) and (31) we have

$$d_2 = \frac{\mathbf{X}'\, d'_2}{\left(\frac{J_{max}}{J'_{max}}\right)^2 \mathbf{X} - \mathbf{X}'}; \quad . \quad . \quad (32)$$

and since $\mathbf{X} = d_1 + d_2$ we have also

$$d_1 = \mathbf{X} - \frac{\mathbf{X}'\, d'_2}{\left(\frac{J_{max}}{J'_{max}}\right)^2 \mathbf{X} - \mathbf{X}'}. \quad . \quad (33)$$

Also we have $\mathbf{X}' = d_1 + d_2 + d'_2$, and $\mathbf{X} = d_1 + d_2$.

Therefore

$$d'_2 = \mathbf{X}' - \mathbf{X}. \quad . \quad . \quad . \quad (34)$$

The value of $\frac{J_{max}}{J'_{max}}$ can be measured off from the two resonance curves, and hence d_1 and d_2 calculated. The following details of a certain decrement determination made as above described will serve as an example. A resonance curve was taken off a certain oscillatory circuit with a cymometer both without and with the cymometer resistance increased by a certain wire. The following readings were taken off the two resonance curves. Find the decrements.

Answer.—

$\frac{J}{J_{max}} = y$	$\left(1 - \frac{n_1}{n_2}\right) = x$	$d_1 + d_2 = \mathbf{X}$
.95	.0120	.115
.90	.0165	.112
.85	.0205	.104
.80	.0255	.107
.75	.0293	.105
.78	.0335	.103
		Mean = .108

From the resonance curve after adding resistance to the cymometer we have

$\frac{J'}{J'_{max}} = y'$	$\left(1 - \frac{n'_2}{n'_1}\right) = x'$	$d_1 + d_2 + d'_2 = \mathbf{X}'$
.95	.0125	.120
.90	.0210	.138
.85	.0255	.130
.80	.0300	.125
.75	.0345	.124
.70	.0385	.119
		Mean = .126

also $\frac{J_{max}}{J'_{max}} = 1.53$. Hence $\left(\frac{J_{max}}{J'_{max}}\right)^2 = 2.34$.

Therefore
$$d_1 + d_2 = 0.108 = \mathbf{X},$$
$$d_1 + d_2 + d'_2 = 0126 = \mathbf{X}'.$$

Hence
$$d'_2 = 0.018,$$

and
$$d_2 = \frac{0.126 \times 0.018}{2.34 \times 0.018 - 0.126} = 0.017,$$
$$d_1 = 0.108 - 0.017 = 0.091.$$

Hence the decrement of the cymometer circuit is $d_2 = 0.017$, and that of the oscillation circuit is $d_1 = 0.091$. Now the frequency n_1 corresponding to resonance was found to be 0.95×10^6. The number of oscillations in a train is therefore

$$\frac{4.605 + 0.091}{0.091} = 51,$$

or 25 complete oscillations per train. The part of the decrement d'_2 due to the added resistance wire was 0.012. Hence that of the cymometer itself is 0.005.

The calculations of decrement are much simplified if in using formula (27) we select such a value of n_2 and n_1 as will make J^2_{max} just twice J^2, since then $\frac{J}{\sqrt{J^2_{max} - J^2}} = 1$. Also if O_1 and O_2 are the values of the oscillation constants of the cymometer corresponding to the maximum current and to the other current which is half the maximum, then the sum of the decrements is given by the expression,

$$\mathbf{X} = d_1 + d_2 = 3.1415 \frac{O_2 - O_1}{O_2}. \qquad (35)$$

If we can then introduce a resistance into the cymometer circuit such that it causes the square of the maximum or resonance current J^2_{max} to drop to half its value equation (31) becomes

$$2\mathbf{X}d_2 = \mathbf{X}'(d_2 + d'_2),$$

or $$(2\mathbf{X} - \mathbf{X}')d_2 = \mathbf{X}'d'_2 ;$$

but $$d'_2 = \mathbf{X}' - \mathbf{X}.$$

Hence $$d_2 = \frac{\mathbf{X}'(\mathbf{X}' - \mathbf{X})}{2\mathbf{X} - \mathbf{X}'}, \quad . \quad . \quad . \quad (36)$$

and $$d_1 = \mathbf{X} - d_2. \quad . \quad . \quad . \quad (37)$$

The equations (36) and (37) then give d_1 and d_2 in terms of the observed quantities $\mathbf{X}$ and $\mathbf{X}'$.

To get good results by this method of Bjerknes using a resonance curve requires a very careful delineation of the curve just at the top part where it reaches a maximum, and this is rather difficult. Appliances have, however, been devised which enable the decrement to be approximately obtained by portable apparatus. Such instruments are called *Decremeters*. See *The Electrician*, vol. lxix. p. 221 (1912), for a portable Decremeter made by Marconi's Wireless Telegraph Company which is constructed as follows :—

It consists of a long single layer coil of wire, AB (see Fig. 27), on which there is a sliding contact E and another sliding or fixed contact C. This coil is made long and of small diameter, so that the fall of potential down it is nearly uniform. It is placed in series with a condenser K of variable capacity. This condenser and inductance forms an oscillatory circuit, in which oscillations are induced by holding it near any other oscillatory circuit to be tested.

A telephone in series with a crystal rectifier is connected between the points C and E, and by sliding E along we can find a point at which the signals are just inaudible. The reciprocal of the length CE along the spiral may then be taken to be proportional to the current J in the circuit. If we vary the tuning by altering the condenser K, the natural frequency n_2 of the circuit is inversely as the square root of the capacity. Hence it is possible to plot a resonance curve in terms of $\mathbf{J} = \frac{1}{S}$ as ordinate and $n_2 = \frac{1}{\sqrt{K}}$ as abscissa, where S = scale length CE and $\sqrt{K}$ = square root of capacity of the condenser. Again, we have an additional inductance which

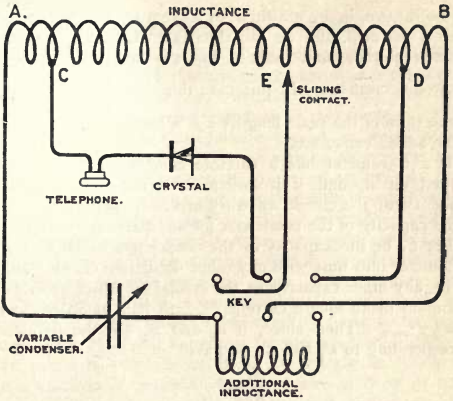


FIG. 27.—Connections in the Marconi Company's Decremeter.

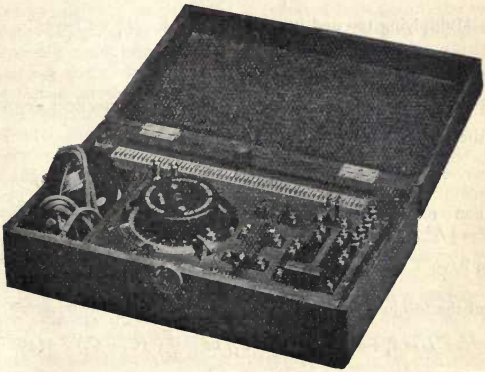


FIG. 28.—External Appearance of the Marconi Decremeter.

can be thrown in by a switch so as to vary the frequency by about 5 per cent, and we can then take the second resonance curve required, and obtain by the above formulæ the required decrement. In this case the ratio $\left(\frac{J'_{max}}{J_{max}}\right)$ is given by the ratio of the scale lengths CE after and before throwing in the added resistance.

If a wavemeter has a variable condenser with capacities marked on it, and if it has a fixed inductance, then the frequency of the circuit will vary inversely as the square root of the capacity of the condenser for any particular setting.

Let C_1 be the capacity of the condensers which brings the cymometer into tune with any other oscillation circuit; and let C_2 be any other capacity of the condenser which reduces the maximum mean-square current J^2_{max} to half its value, so that $J^2 = \frac{1}{2}J^2_{max}$. Then since, if n_1 and n_2 are the frequencies corresponding to C_1 and C_2 we have

$$\frac{n_2}{n_1} = \frac{\sqrt{C_1}}{\sqrt{C_2}}.$$

Hence

$$1 - \frac{n_2}{n_1} = \frac{\sqrt{C_2} - \sqrt{C_1}}{\sqrt{C_2}}$$

Multiplying top and bottom by $\sqrt{C_2} + \sqrt{C_1}$ we have

$$1 - \frac{n_2}{n_1} = \frac{C_2 - C_1}{C_2 + \sqrt{C_1 C_2}}.$$

If $\sqrt{C_1}$ and $\sqrt{C_2}$ do not differ by more than 5 per cent we can write this last equation in the form

$$1 - \frac{n_2}{n_1} = -\frac{C_2 - C_1}{2C_2}. \quad . \quad . \quad . \quad (38)$$

Then referring to the formula (27) we see that when $J^2 = \frac{1}{2}J^2_{max}$

we have

$$\frac{J}{\sqrt{J^2_{max} - J^2}} = 1, \quad . \quad . \quad . \quad (39)$$

and the formula for the total semiperiod decrement becomes

$$D = d_1 + d_2 = \frac{\pi}{C_2 2}(C_2 - C_1) \text{ or } = \frac{\pi}{2C_1}(C_1 - C_2). \quad (40)$$

depending on whether C_1 or C_2 is largest. The decrement is therefore proportional to the change in capacity required to reduce the wave meter mean-square current to half its maximum value.

4. Energy Losses in Condensers

This matter has already been briefly considered in Chapter V. It has there been shown that the energy losses in a condenser, from whatever cause arising, whether dielectric conductivity or hysteresis, can be represented as if they were due to a resistance of a certain magnitude placed in series with a condenser having a perfect dielectric with no energy loss in it.

Hence, if we take an air-condenser as approximating to a perfect condenser we can put in series with it an adjustable non-inductive resistance, and vary this resistance until the combination represents exactly some other condenser having dielectric losses. Owing to the difficulty of obtaining air-condensers of variable capacity which is at the same time very large, this method is only adapted for tests made with condensers of comparatively small capacity such, for instance, as are used in receiving apparatus for wireless telegraphy.

To put it in practice we have to provide an oscillatory circuit, which may consist of a coil of wire of 8 or 10 turns in series, with one or more Leyden jars and with a quenched spark discharger. This last is best of the form devised by Peukert and by the author consisting of a fixed metal plate and one parallel and very close to it revolved by a motor, the two plates being immersed in paraffin oil. Such a spark gap gives a nearly dead beat or quenched spark discharge. Parallel and not very close to the above primary coil of 8 or 10 turns is placed another similar coil, which has in series with it a thermoelectric ammeter and a variable resistance consisting of two fine high-resistance wires stretched over a strip of ebonite. These wires can be short-circuited by a slider so as to vary their total resistance. The condenser under test, and a variable air-condenser having a somewhat greater capacity, are placed so as to be alternately switched in to the secondary circuit. We begin observations by putting in the condenser under test whose dielectric losses are required, and we short-circuit, or cut out all the variable resistances. We then set going the primary oscillations, and take a reading of the ammeter in the secondary circuit. It is desirable to be able to vary the inductance in the secondary circuit so as to tune the secondary circuit to the primary.

We next substitute for the condenser under test the variable air-condenser, and first adjust its capacity so that the thermoelectric ammeter gives a maximum reading, and then insert more or less of the sliding resistance until the

maximum current reading has the same absolute value as in the case of the condenser under test. We then determine this resistance R, and the absolute capacity C of the equivalent air-condenser, and the value in milli-amperes of the current A through the condenser.

We can then say that the energy loss in the condenser under test is equal to A^2R milliwatts, and that the *power-factor* of the condenser or cosine of the angle of phase difference of current and voltage is nearly $CR\rho$, where $\rho = 2\pi n$ and n is the frequency.

Also the terminal voltage of the condenser under test is $\frac{A}{C\rho}$ millivolts. As an illustration of a set of determinations of this kind on various condensers we give the figures from measurements made in the author's laboratory.

Frequency. n.	Capacity in mfds. C.	Current in Milli-Amperes. A.	Equivalent Resistance in Ohms. R.	Condenser P.D. in Volts. V.	Power-Factor. $P.F. = CR\rho$.
I. Ebonite Plate Condenser					
0.64×10^6	0.000865	44.3	3.35	12.9	0.012
	0.001650	59.5	1.60	9.0	0.011
	0.002495	55.0	1.22	5.5	0.012
	0.003760	71.0	0.85	4.7	0.013
	0.004230	68.0	0.65	4.0	0.011
	0.005323	72.0	0.50	3.4	0.011
1.48×10^6	0.000715	65.5	1.48	9.8	0.010
	0.001615	66.5	0.99	4.4	0.015
	0.002510	72.0	0.55	3.1	0.013
	0.002870	67.0	0.37	2.5	0.010
	0.003175	65.0	0.33	2.2	0.010
II. Glass Plate Condenser					
1.49×10^6	0.00046	65.5	1.28	15.4	0.005
1.90×10^6	0.00046	67.0	0.87	12.2	0.005
III. Moscicki Condenser					
0.64×10^6	0.00174	68.0	1.69	9.7	0.012
0.92×10^6	,,	68.0	1.52	6.8	0.015
1.43×10^6	,,	72.0	1.28	4.6	0.020

The energy loss in milliwatts is given by the product of A, V, and $P.F.$

CHAPTER VIII

AERIALS AND ELECTRIC RADIATION

1. Maxwell's Theory of the Electromagnetic Field

THE two fundamental laws of electromagnetism are (i.) the law of Ampère, which tells us the relation between a current in a conductor and the magnetic field round it; and (ii.) the law of Faraday, which tells us the relation between the variation of the magnetic field passing through a circuit and the electromotive force induced in it.

If C is the current in a circuit and H the magnetic force at any point outside it, then Ampère's law mathematically stated is that, *The line integral of magnetic force round any closed line embracing the circuit is equal numerically to* 4π *times the total current flowing in the circuit.* Faraday's law is: *The line integral of electric force round any circuit created by variation of magnetic flux through it is numerically equal to that rate of change of flux.*

We have seen in Chap. I. that line integrals of vectors taken round unit area in a vector field are called the *curls* of those vectors.

Hence if C is current density or current through unit area, and H and E are magnetic and electric forces, and μ and K are the magnetic permeability and dielectric coefficients respectively of the medium, then the above laws stated symbolically are

$$4\pi C = Curl\ H, \quad . \quad . \quad . \quad (1)$$

$$-\mu\dot{H} = Curl\ E, \quad . \quad . \quad . \quad (2)$$

where $\dot{H}$ stands for $\frac{dH}{dt}$, or time rate of change of magnetic force.

These equations had long been familiar to electricians as

applied to circuits of conductors or metal wires, but Maxwell's great contribution to electromagnetism was that he saw they applied also to circuits formed of dielectrics, or so-called insulators, but that in this case the current C was not a *conduction current* but a *displacement current*, or rate of change of *electric strain* or *displacement*.

If D denotes this strain, displacement, or electric flux, and K is the dielectric coefficient, then Maxwell shows that

$$D = \frac{K}{4\pi} E,$$

and hence

$$\frac{dD}{dt} = \dot{D} = \frac{K}{4\pi} \dot{E}. \quad . \quad . \quad . \quad (3)$$

Accordingly the equations (1) and (2) then become

$$K\dot{E} = Curl\ H, \quad . \quad . \quad . \quad (4)$$

$$-\mu\dot{H} = Curl\ E. \quad . \quad . \quad . \quad (5)$$

If we differentiate (5) with respect to time, and substitute in (5) the value of E from (4), we have

$$-\mu K\ddot{H} = Curl^2\ H,$$

where $\ddot{H}$ stands for $\frac{d^2H}{dt^2}$.

But now if there are no magnetic poles in the region, the vector H is a non-divergent vector, and hence by the theorem given in Chap. I. § 16, eq. (160), we have

$$-Curl^2 H = \nabla^2 H = \frac{d^2H}{dx^2} + \frac{d^2H}{dy^2} + \frac{d^2H}{dz^2}.$$

Accordingly then

$$\frac{d^2H}{dt^2} = \frac{1}{\mu K}\left(\frac{d^2H}{dx^2} + \frac{d^2H}{dy^2} + \frac{d^2H}{dz^2}\right), \quad (6)$$

and a similar equation can be found with E instead of H.

This equation is the general equation of *wave motion*, and it shows that E and H, viz. the electric and magnetic forces, do not make their appearance suddenly at any distant point, but are propagated through space as a wave motion with a speed equal to $\frac{3 \times 10^{10}}{\sqrt{\mu K}}$ centimetres per second, or $\frac{3 \times 10^{5}}{\sqrt{\mu K}}$ kilometres per second.

Thus, for example, in air, for which $\mu = 1$ and $K = 1$, the

wave velocity is 3×10^{10} cms. per second, or 300,000 kilometres per second.

In glass, for which $\mu = 1$ but $K = 9$, the velocity would be only 100,000 kilometres per second; and in sulphur, for which $\mu = 1$, $K = 4$, the velocity would be 150,000 kilometres per second.

Maxwell showed that the electric and magnetic forces E and H are related to the scalar and vector potentials V and A, in the manner expressed by the following vector equations:—

$$\textit{Electric force} = -\left(\begin{matrix}\textit{The gradient of the}\\ \textit{scalar potential}\end{matrix}\right) - \left(\begin{matrix}\textit{The time rate}\\ \textit{of change of the vector}\\ \textit{potential}\end{matrix}\right). \quad (7)$$

$$\textit{Magnetic force} = \textit{curl of vector potential}. \quad (8)$$

These equations, expressed in the vector notation described in Chap. I. § 16, are

$$\mathbf{E} = -\nabla V - \dot{\mathbf{A}}, \quad . \quad . \quad . \quad (9)$$

$$\mathbf{H} = \nabla \times \mathbf{A}. \quad . \quad . \quad . \quad (10)$$

The reader should remember that the cross ($\times$) between vector quantities expressed in thick type denotes vector multiplication (see Chap. I.).

If the rectangular components of electric force $\mathbf{E}$ are denoted by X, Y, and Z, and those of magnetic force $\mathbf{H}$ by α, β, γ, and those of vector potential $\mathbf{A}$ by F, G, and H, then we have

$$\left.\begin{aligned} X &= -\frac{dV}{dx} - \frac{dF}{dt} \\ Y &= -\frac{dV}{dy} - \frac{dG}{dt} \\ Z &= -\frac{dV}{dz} - \frac{dH}{dt} \end{aligned}\right\} \quad . \quad . \quad . \quad (11)$$

$$\left.\begin{aligned} \alpha &= \frac{dH}{dy} - \frac{dG}{dz} \\ \beta &= \frac{dF}{dz} - \frac{dH}{dx} \\ \gamma &= \frac{dG}{dx} - \frac{dF}{dy} \end{aligned}\right\} \quad . \quad . \quad . \quad (12)$$

The equations (1) to (12) inclusive comprise Maxwell's theory of the electromagnetic field.

If at any place there is an element of electric quantity or

charge dq, then at a point at a distance r the scalar potential $V=\frac{dq}{r}$.

If at any point there is an element of current ids, then at a point at a distance r the component of the vector potential parallel to the element of length ds is $\frac{ids}{r}$. Hence we can find the electric and magnetic forces from equations (11) and (12) or (9) and (10); and therefore delineate the field round any simple form of oscillator.

2. Theory of the Hertzian Oscillator

Hertz introduced a new form of oscillatory circuit, consisting of two metal balls or spheres attached to the ends of a short metal rod. This rod was cut in the centre and a small air gap left between the two parts. If the two parts are connected to an induction coil and charged to different potentials, then, corresponding to a certain value of the potential, a spark passes across the gap and connects the two sides. Then the opposite electric charges on the balls rush together, and electric oscillations ensue. If Q is the maximum value of the charge + or − on each ball before discharge, and if l is the length of the rod, then the product $Ql=\phi$ is called the *electric moment* of the oscillator. We can only deal with the problem by simple mathematics if we assume that the current is the same in all parts of the rod at the same instant. This means that the capacity of the balls is large compared with that of the rod.

The free electric charge which exists at any moment on the balls will create at distant points a scalar potential, and the electric current in the rod, which increases as the balls charge or discharge, creates a vector potential which has direction in space parallel to the rod. If the charge q on the balls varies as a sine function of the time t, it can be represented by the formula $q=Q\ sin\ nt$, where $n=\frac{2\pi}{T}$ and T is the periodic time of the oscillation.

Now the potential due to the periodically varying charge q at a point at a distance r is not $\frac{q}{r}=\frac{Q\ sin\ nt}{r}$, but must be denoted by $\frac{Q\ sin\ (mr-nt)}{r}$, where $m=\frac{2\pi}{\lambda}$, because the potential

takes time to reach the distant point and travels with a speed $W=\frac{\lambda}{T}=\frac{n}{m}$.

If we denote $mr-nt$ by χ, and

$$\frac{sin\ (mr-nt)}{r} \text{ by } \Pi, \quad . \quad . \quad (13)$$

and if we consider that the axis of the oscillator coincides with the axis of y, then the expressions for the scalar potential V and vector potential G are

$$\left.\begin{aligned} V&=-\phi\frac{d\Pi}{dy}\\ G&=\frac{\phi}{c}\frac{d\Pi}{dt}\end{aligned}\right\} \quad . \quad . \quad . \quad (14)$$

where $c=\frac{n}{m}=$ velocity of wave, and G is the component of the vector potential, parallel to the axis of the oscillator.

Hence it can be shown (see J. A. Fleming, *Principles of Electric Wave Telegraphy and Telephony*, chap. v.) that at a distance r from the oscillator, large compared with the length of the oscillator, the components of the electric and magnetic forces in space are given by the following expressions.

Inserting the values of V and G from (14) in the equations (11) and (12), we have for the components X, Y, Z, of the electric force, and α, β, γ, of the magnetic force, the values

$$\left.\begin{aligned} X&=\phi\frac{d^2\Pi}{dxdy} & \alpha&=-\frac{\phi}{c}\frac{d^2\Pi}{dzdt}\\ Y&=-\phi\left\{\frac{d^2\Pi}{dx^2}+\frac{d^2\Pi}{dz^2}\right\} & \beta&=0, \quad . \\ Z&=\phi\frac{d^2\Pi}{dydz} & \gamma&=\frac{\phi}{c}\frac{d^2\Pi}{dxdt};\end{aligned}\right. \quad . \quad (15)$$

and inserting the value of Π from (13), and considering only large distances from the oscillator, we have

$$\text{Components of E} \left\{\begin{aligned} X&=-\frac{\phi m^2}{r}\ sin\ \chi\ sin\ \theta\ cos\ \theta,\\ Y&=\frac{\phi m^2}{r}\ sin\ \chi\ \frac{x^2+z^2}{r^2}.\end{aligned}\right. \quad (16)$$

$$\text{Components of H} \left\{\begin{aligned} \alpha&=-\frac{\phi mn}{cr}\ sin\ \chi\ \frac{z}{r},\\ \gamma&=\frac{\phi mn}{cr}\ sin\ \chi\ sin\ \theta,\end{aligned}\right. \quad (17)$$

where χ is written for $(mr - nt)$, and θ is the angle between the direction of r and the axes of the oscillator, so that $\frac{x}{r} = \sin\theta$ and $\frac{y}{r} = \cos\theta$.

Hence the lines of magnetic force round the oscillator are circles with centres on the axes of the oscillator, and the lines of electric force are closed loops in planes passing through the axes of the oscillator (see J. A. Fleming, *Principles of Electric Wave Telegraphy and Telephony*, chap. v. 3rd ed.).

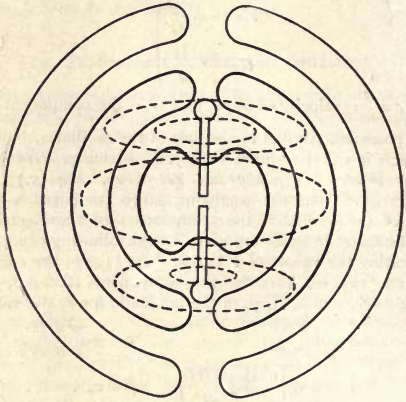


FIG. 29.—Rough Diagram of Field round a Hertzian Oscillator.

At large distances the forces are therefore inversely as the distance from the oscillator. In Fig. 29 are shown approximately the form of the lines of force round a Hertzian oscillator, the electric force lines being drawn in firm lines and the magnetic lines in dotted lines.

3. Radiation from a Hertzian Oscillator. Poynting's Theorem

In an electromagnetic field there is energy present in two forms, electric and magnetic. If K is the dielectric coefficient and μ the magnetic permeability of the medium, then at any

point where the electric force is E and the magnetic force H, the electrostatic energy per unit of volume of the medium is $\frac{1}{8\pi}KE^2$ and the magnetic energy is $\frac{1}{8\pi}\mu H^2$ per unit of volume. Hence the rate at which energy is being supplied to the field is

$$\frac{d}{dt}\left(\frac{1}{8\pi}KE^2\right)+\frac{d}{dt}\left(\frac{1}{8\pi}\mu H^2\right)=\frac{K}{4\pi}E\frac{dE}{dt}+\frac{\mu}{4\pi}H\frac{dH}{dt}, \quad (18)$$

provided we consider the electric displacement and magnetic flux are in the same direction as E and H.

If we write $\frac{dW}{dt}$ for the rate at which energy is supplied to the field, then even if the displacement and flux is not coincident in direction with the forces, we can write (18) in the form of a vector equation

$$4\pi\frac{dW}{dt}=\mathbf{E}.(K\dot{\mathbf{E}})+\mathbf{H}.(\mu\dot{\mathbf{H}}), \quad . \quad . \quad (19)$$

where the *dot* over the letter signifies differentiation with regard to time, and the *dot* between the letters signifies scalar multiplication.

But now by equations (4) and (5)

$$K\dot{\mathbf{E}}=Curl\ H=\nabla\times\mathbf{H},$$

and

$$\mu\dot{\mathbf{H}}=Curl\ E=-\nabla\times\mathbf{E}.$$

Hence (19) becomes

$$4\pi\frac{dW}{dt}=\mathbf{E}.(\nabla\times\mathbf{H})-\mathbf{H}.(\nabla\times\mathbf{E}), \quad . \quad (20)$$

or,

$$4\pi\frac{dW}{dt}=\left(\begin{matrix}Scalar\ product\ of\\ electric\ force\ and\\ curl\ of\ magnetic\\ force\end{matrix}\right)-\left(\begin{matrix}Scalar\ product\ of\\ magnetic\ force\\ and\ curl\ of\\ electric\ force\end{matrix}\right). \quad (21)$$

By a well-known vector theorem this last equation can be written

$$4\pi\frac{dW}{dt}=\nabla.(\mathbf{E}\times\mathbf{H}). \quad . \quad . \quad (22)$$

Let $\frac{\mathbf{E}\times\mathbf{H}}{4\pi}$ or $\frac{EH\sin\theta}{4\pi}=\mathbf{R}$, where θ is the angle between $\mathbf{E}$ and $\mathbf{H}$. Then $\mathbf{R}$ is called the radiant vector, and the rate at which energy is appearing per unit of volume in the medium is the *divergence* of $\mathbf{R}$. By the *divergence theorem* (see Chap. I. § 18) this can be expressed as the surface integral of $\mathbf{R}$,

and hence we have the following important theorem, called Poynting's Theorem :—

The rate at which energy is passing through unit area of a closed surface described in the electromagnetic field is equal to the surface integral of the radiant vector taken all over the surface, or to $\frac{1}{4\pi}$ times the vector product of **E** and **H** integrated over the surface.

To apply this to the Hertzian oscillator, we have at large distances in this case **E** and **H**, at right angles to each other and to the radius vector from the centre of the oscillator, and hence in electrostatic units we have,

$$\left.\begin{aligned} \mathbf{E} &= \frac{\phi m^2}{r} \sin \chi \sin \theta \\ \mathbf{H} &= \frac{\phi m m}{r} \sin \chi \sin \theta \end{aligned}\right\} \quad . \quad . \quad . \quad (23)$$

Hence $$\mathbf{R} = \frac{\mathbf{E} \times \mathbf{H}}{4\pi} = \frac{\phi^2 m^2 n}{4\pi r^2} \sin^2 \chi \sin^2 \theta. \quad . \quad (24)$$

Integrating this over the surface of a sphere of radius r we have for the energy sent out in ergs per period

$$R = \frac{16}{3} \frac{\phi^2 \pi^4}{\lambda^3}, \quad . \quad . \quad . \quad (25)$$

where ϕ is the electric moment of the oscillator, and λ is the wave-length of the radiation, or $2\pi/m$.

This expression (25) gives us the radiation in *ergs per complete period of the oscillator.*

It is in practice more convenient to express the mean radiation per second in watts, and in place of the electric moment ϕ to substitute the maximum or mean-square value of the electric current in the centre of the oscillator.

Let C be the capacity of each sphere or half of the oscillator with respect to the other, and let V be the potential difference before discharge, then CV is the charge on each half, and $CVl = \phi$ where l is the length of the oscillator.

If C is measured in microfarads and V in volts, then since $C \times 9 \times 10^5$ is the capacity in electrostatic units, and since $\frac{V}{300}$ is the potential difference in electrostatic units, we have then $\phi = 3000\ CVl$.

Again, if A is the maximum value of the current in amperes at the centre of the oscillator, we have $A = \frac{nCV}{10^6}$, where $n = 2\pi N$ and N is the frequency. Hence we have

$$\phi = \frac{3 \times 10^9 lA}{n},$$

and hence in ergs per period

$$R = \frac{4\pi^2 10^8 l^2 A^2}{\lambda^2 N}, \quad . \quad . \quad . \quad (26)$$

and the radiation in watts is

$$P = 40\pi^2 \frac{l^2}{\lambda^2} A^2. \quad . \quad . \quad . \quad (27)$$

If we denote the root-mean-square (R.M.S.) value of the current at the centre of the oscillator by a, then since $A^2 = 2a^2$ we have from (27)

$$P = 80\pi^2 \frac{l^2}{\lambda^2} a^2, \quad . \quad . \quad . \quad (28)$$

or since $\pi^2 = 9.87$

$$P = 789.6 \frac{l^2}{\lambda^2} a^2. \quad . \quad . \quad . \quad (29)$$

This gives us the radiation in watts from a Hertzian oscillator of length l sending out waves of wave-length λ.

The ratio $\frac{l}{\lambda}$ is a fixed ratio, and for a Hertzian linear oscillator is not far from 0.5, hence $l^2/\lambda^2 = 0.25$ and in this case

$$P = 197.4a^2.$$

Hence the radiation is independent of everything but the current at the centre of the oscillator.

For any oscillator the radiation in watts or joules per second is given by an expression of the form

$$P = C\frac{l^2}{\lambda^2} a^2 = R_a a^2, \quad . \quad . \quad (30)$$

where C is some numerical constant. Hence P is proportional to the mean square value of the current at the centre reckoned in amperes and to a certain constant R_a called the *radiation resistance* of the antenna.

Just as the ohmic resistance of a circuit is a quantity by which we multiply the square of the current in it to obtain

the energy dissipation in it, so the radiation resistance of an oscillator is the quantity by which we multiply the current at the centre or base to obtain the energy radiated per second from it.

4. Radiation from Wireless Telegraph Aerials or Antennæ

The foregoing formulæ have to be modified in the case of aerial wires or antennæ which are earthed at the base.

An earthed Marconi aerial may be regarded as a Hertzian oscillator half-buried in the earth. The buried half does not radiate. Let h be the height above ground of an aerial wire or wireless telegraph antenna, then $2h$ is the length of the equivalent Hertzian oscillator. Consider the case of a flat top or T antenna in which we can fairly well assume that the capacity of the aerial is chiefly collected at the top. Hence if h is the height of the vertical part of the T, we can consider that this case is analogous to a Hertzian oscillator as above discussed of length $l = 2h$, but of which one-half does not radiate.

We have shown that for such a Hertzian oscillator the radiation in watts is by (29)

$$P = 789.6 \frac{l^2}{\lambda^2} a^2. \quad . \quad . \quad . \quad (31)$$

Now substitute $2h$ for l and take half of the result and we have

$$P = 1578.2 \frac{h^2}{\lambda^2} a^2. \quad . \quad . \quad . \quad (32)$$

Accordingly (32) gives us the radiation for a flat top or T aerial of vertical height h.

Example.—A wireless telegraph antenna consists of a set of vertical wires 200 feet in height, with long horizontal wires parallel to the earth's surface attached to the top. The wave-length of the emitted radiation is 10,000 feet, and the current as measured by a hot wire ammeter inserted at the base near the earth plate is 50 amperes. Find the energy radiation.

Answer.—$h = 200$ feet, $\lambda = 10{,}000$ feet, $a = 50$ amperes; then by (32)

$$P = 1578.2 \left(\frac{200}{10000}\right)^2 (50)^2 = 1.578 \text{ kilowatts},$$

or nearly 2-horse power.

This power all goes off as electric waves. In addition there

is an energy waste in the antenna due to its high frequency resistance which can be calculated when the wire of which it is made is given.

Definition.—The *radiation efficiency* of an antenna is the ratio expressed as a percentage of the energy radiated by it to the total energy given to it.

Consider in the next place a plain single wire antenna or aerial. In this case the capacity is not located at the top entirely, but is distributed all along the wire. If h is the total height of the aerial then it can be shown that the antenna has an electric movement the same that it would have if all its actual distributed capacity were collected at a point at a height $\frac{2}{\pi}h$ above the surface of the earth. Now the double of this, viz. $\frac{4}{\pi}h$, is then the length of the equivalent Hertzian oscillator, of which only one-half radiates.

Hence the radiation in watts from the single wire or plain aerial is

$$P = \frac{1}{2}789.6\left(\frac{\frac{4}{\pi}h}{\lambda}\right)^2 a^2, \quad . \quad . \quad (33)$$

$$= 789.6\frac{8}{\pi^2}\cdot\frac{h^2}{\lambda^2}a^2,$$

or

$$P = 640\frac{h^2}{\lambda^2}a^2. \quad . \quad . \quad . \quad (34)$$

This gives the radiation in watts for a plain aerial of height h.

Example.—A single vertical wire aerial 150 feet high emits electric waves 750 feet in length. The current at the base of the wire as read by a hot wire ammeter is 5 amperes. What is the radiation?

Answer.—$h = 150$ feet, $\lambda = 750$ feet, $a = 5$ amps.

$$P = \frac{640}{25} \times 25 = 640 \text{ watts},$$

$$= 0.64 \text{ kilowatt}.$$

5. The Radiation Decrement of an Oscillator

Any source of energy dissipation in an oscillatory circuit such as resistance in the conductor or conductance in the dielectric damps out free oscillations set up in it. Hence

since radiation sends away energy from the oscillator in the form of electric wave, radiation itself is a cause of damping, and the corresponding decrement is called the radiation decrement. This can be determined from the expression (25) already given for the energy loss in ergs per complete period of oscillation.

For the energy loss per semiperiod is $\frac{8\pi^4\phi^2}{3\lambda^3}$ and $\phi = CVl$, where C is the capacity of one-half of the oscillator with respect to the other and V is the P.D. of the two parts.

If I_1 is the amplitude of the first oscillation and I_2 that of the next in the opposite direction, then if δ is the decrement per semiperiod, we have by definition

$$\delta = log_\epsilon \frac{I_1}{I_2}, \quad . \quad . \quad . \quad (35)$$

or

$$\frac{I_2}{I_1} = \epsilon^{-\delta}. \quad . \quad . \quad . \quad (36)$$

Hence

$$\left(\frac{I_2}{I_1}\right)^2 = \epsilon^{-2\delta}. \quad . \quad . \quad . \quad (37)$$

Now the energy is proportional to the square of the amplitude. Hence the loss of energy between the two oscillations or in one semiperiod is to the energy in the first oscillation in the ratio

$$\frac{I_1^2 - I_2^2}{I_1^2} = 1 - \epsilon^{-2\delta} = 2\delta \text{ nearly when } \delta \text{ is small.}$$

The energy originally given to the oscillator is $\frac{1}{2}CV^2$. Hence we have

$$\frac{I_1^2 - I_2^2}{I_1^2} = \frac{8\pi^4 l^2 C^2 V^2}{3\lambda^3 \frac{1}{2} CV^2} = \frac{16\pi^4 l^2 C}{3\lambda^3} = 2\delta, \quad . \quad (38)$$

or

$$\delta = \frac{16\pi^4 l^2 C}{6\lambda^3}, \quad . \quad . \quad . \quad (39)$$

where C is in electrostatic units.

Example.—Hertz constructed an oscillator by placing two metal spheres each 15 cms. in radius at the ends of a metal rod 100 cms. long. In the centre of the rod was a spark gap. He found the wave-length of the radiation emitted was 560 cms. Find the radiation decrement and state how many complete oscillations will take place before the extinction of each train.

Answer.—The capacity of a sphere in free space in electrostatic units is equal to its radius in centimetres, but the

capacity of two equal spheres with respect to each other is equal to half of the radius of each because the two separate capacities are then in series. Hence in formula (39) we have $C = 7.5$, also $\pi^4 = 97.4$, $l = 100$ cms., $\lambda = 560$ cms. Hence

$$\delta = \frac{16 \times 97.4 \times (100)^2 \times 7.5}{6 \times (560)^3} = 0.11.$$

This is the decrement per semiperiod.

To find how many oscillations take place we must know the original energy storage. This is equal to $\frac{1}{2}CV^2$, where V is the spark potential just before discharge. Suppose the spark gap is 5 mm. in length, then this corresponds to about 15,000 volts or to 50 E.S. units. Hence in this case the energy stored before discharge is $\frac{1}{2} \times 7.5 \times (50)^2 = 9375$ ergs. The energy radiated per complete period is

$$\frac{16\pi^4\phi^2}{3\lambda^3} = \frac{16\pi^4}{3\lambda^3} C^2V^2l^2 = \frac{1600 \times (7.5 \times 50 \times 100)^2}{3 \times 175.6 \times 10^6} \text{ ergs}$$

$$= 4200 \text{ ergs nearly.}$$

Hence all the stored energy is radiated in about two complete oscillations.

If the high frequency resistance of the rod or circuit of an oscillator is R' and its inductance L, and if the frequency of the oscillations is N, then the resistance semiperiod decrement is $\frac{R'}{4NL}$.

Hence the total decrement D of an oscillator due partly to its resistance and partly to radiation is

$$D = \frac{R'}{4NL} + \frac{16\pi^4l^2C}{3\lambda^3}, \quad . \quad . \quad (40)$$

or

$$D = \frac{3 \times 10^{10}R'C\pi^2}{\lambda} + \frac{16\pi^4l^2C}{3\lambda^3}. \quad . \quad (41)$$

In the case of the oscillator above mentioned the rod was 5 mm. in diameter and 100 cms. long, and hence its inductance L was about 1134 cms. Also the frequency N was about 55×10^6. Hence $4NL$ is 250×10^9 or 250 ohms.

The oscillator was charged by a spark gap in the centre, and hence the resistance R' was chiefly that of the spark, which might be about 5 ohms. Hence the resistance decrement is $\frac{5}{250}$ or about 0.02.

Accordingly the total decrement would be $0.02 + 0.11 = 0.13$. The radiation decrement is ten times greater than the

resistance decrement, and this shows that the energy dissipation by radiation for such an open oscillator is vastly greater than that due to its resistance.

It should be noted that any glow discharge or brush discharge from the aerial wires involves dissipation of energy which has to be reckoned in to the joulean or frictional resistance losses. The value of it is difficult to determine.

6. Radiation from Closed or Semiclosed Oscillators

Since most forms of antenna used in wireless telegraphy are combinations of open and closed radiators, we have to consider the theory of a completely closed oscillator. The simplest form is constructed of four overlapping Hertzian oscillators in the shape of a square. Let I be the value of the current flowing round the circuit. Then if S is the area of the oscillator, $IS = M$ is called the *magnetic moment.* Since there are no free electric poles, there is no scalar potential at external points, but there is a vector potential whose components are parallel to the sides of the square. Since the current I may be considered as equal to Qn where Q is the maximum charge on one Hertzian oscillator, the quantity $\frac{M}{cn}$ takes the place of ϕdy in the expression for the vector potential.

Hence if the plane of the closed square oscillator is the zx plane, and if the area of the oscillator is $\delta z . \delta x$, then the components of the vector potential are

$$F = -\frac{M}{cn}\frac{d\dot{\Pi}}{dz}, \quad H = \frac{M}{cn}\frac{d\dot{\Pi}}{dx}, \quad . \quad . \quad (42)$$

where $$\dot{\Pi} = \frac{d\Pi}{dt} \text{ and } \Pi = \frac{\sin(mr - nt)}{r}$$

as before. Hence the expressions for the components of the electric and magnetic forces by equations (11) and (12) are

$$X = \frac{M}{cn}\frac{d^2\dot{\Pi}}{dzdt} \qquad \alpha = \frac{M}{n}\frac{d^2\dot{\Pi}}{dxdy}$$

$$Y = 0 \qquad \beta = -\frac{M}{n}\left\{\frac{d^2\dot{\Pi}}{dx^2} + \frac{d^2\dot{\Pi}}{dz^2}\right\}. \quad (43)$$

$$Z = -\frac{M}{cn}\frac{d^2\dot{\Pi}}{dxdt} \qquad \gamma = \frac{M}{n}\frac{d^2\dot{\Pi}}{dydz}.$$

On comparing these with equation (15) for the open oscillator it will be seen that they differ in having $\dot{\Pi}$ for Π and electric and magnetic forces are interchanged. In the above equations c stands for the conversion factor 3×10^{10} for converting electrostatic to electromagnetic units.

Hence since

$$\dot{\Pi} = -\frac{n \cos(mr - nt)}{r} = \frac{-n \cos \chi}{r}$$

we have

$$X = -\frac{M}{c}\frac{mn}{r}\cos\chi\frac{z}{r} \qquad \alpha = -\frac{Mm^2}{r}\cos\chi\frac{x}{r}\cdot\frac{y}{r}$$

$$Y = 0 \qquad \beta = \frac{Mm^2}{r}\cos\chi\frac{x^2+z^2}{r^2} \cdot \quad (44)$$

$$Z = \frac{M}{c}\frac{mn}{r}\cos\chi\frac{x}{r} \qquad \gamma = \frac{Mm^2}{r}\cos\chi\frac{y}{r}\frac{z}{r}$$

If we take a point P in the plane xy which is perpendicular to the plane of the closed oscillator, then the electric force at $P = Z = \frac{Mmn}{cr}\cos\chi\frac{x}{r}$ and the magnetic force at P is the resultant of α and β perpendicular to the radius vector r and is $-\frac{Mm^2}{r}\cos\chi\frac{x}{r}$. Hence by Poynting's Theorem the energy sent out in a time dt through an area dS of a circumscribing sphere is

$$dW = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H} dS dt. \qquad . \quad . \quad (45)$$

If we take dS to be an elementary zone of angular width $d\theta$, then $dS = 2\pi r^2 \sin\theta d\theta$, and since $\frac{x}{r} = \sin\theta$ we have

$$dW = \frac{M^2 n m^2}{2}\cos^2\chi \sin^3\theta d\theta dt, \qquad . \quad (46)$$

and integrating this from $\theta = 0$ to $\theta = \frac{\pi}{2}$ and from $t = 0$ to $t = T$ we have

$$W = \frac{16\pi^4 M^2}{3\lambda^3}, \qquad . \quad . \quad . \quad (47)$$

which gives us the energy in ergs sent out by the oscillator per complete period.

If we put A for the maximum value of the current in amperes during the period, then since the absolute electromagnetic unit of currents is 10 amperes, we have

$$W = \frac{16\pi^4 S^2 A^2}{300\lambda^3} \left(\begin{matrix} \text{in ergs per} \\ \text{period} \end{matrix} \right). \quad . \quad . \quad (48)$$

But $16\pi^4 = 1558$ nearly. Hence

$$W = \frac{5 S^2 A^2}{\lambda^3} \left(\begin{matrix} \text{in ergs per} \\ \text{period} \end{matrix} \right). \quad . \quad . \quad (49)$$

If N denotes frequency, then since 10^7 ergs equal 1 joule, and since $N\lambda = 3 \times 10^{10}$, we have

$$P = \frac{S^2 N^4 A^2}{54 \times 10^{36}} \left(\begin{matrix} \text{in joules per second} \\ \text{or watts} \end{matrix} \right), \quad . \quad (50)$$

which gives us the energy radiated per second in joules.

From equation (27) it is seen that for an open or Hertzian oscillator the radiation in joules per second or watts is

$$P = \frac{40\pi^2 l^2 A^2 N^2}{9 \times 10^{20}},$$

$$= \frac{43.8 l^2 N^2 A^2}{10^{20}}. \quad . \quad . \quad (51)$$

Hence comparing expressions (50) and (51) it is clear that the radiation is not only much greater for the open oscillator for a given current, but whilst it increases only as the square of the frequency for the open oscillator, it increases as the 4th power of the frequency for the closed oscillator. Therefore, for low frequency oscillations the closed oscillator hardly radiates to any sensible extent. A closed oscillatory circuit is therefore sometimes called a non-radiative circuit; but it is not so for very high frequency oscillations.

A semiclosed antenna, such as a Marconi directive aerial, or a nearly closed Bellini and Tosi aerial, may be regarded as the sum of a closed and an open antenna.

If a closed square circuit has placed closely adjacent to one side a linear oscillator equal in length, and if oscillations are set up in both the closed and open circuit, but so that they are in opposite directions in the open and adjacent side of the closed circuit, the electrical effect at a distance will be the sum of the effects of the open and closed oscillators.

The electromagnetic field of the open oscillator is symmetrical all around it, and that of the closed oscillator has a field which is symmetrical with reference to an axis at right

angles to its plane. Hence the field of the open oscillator strengthens that of the closed on one side of the closed oscillator and weakens it on the other in the plane containing the closed oscillator.

It follows therefore that a bent oscillator forming three sides of a square or rectangle has an unsymmetrical radiation and radiates most strongly in a direction opposite to that in which its free ends point. If we consider a symmetrical plane drawn through the centre of such an oscillator, and at right angles to its own plane, we can then represent by the radii of a polar curve the radiation in different directions. Marconi has shown experimentally that if an aerial is formed consisting of a short vertical part and a much longer horizontal part, the radiation from it when oscillations are set up in it is unsymmetrical and greatest in the direction opposite to that in which the free end points. The radiation in horizontal directions in different azimuths is represented by the radii of a polar curve resembling a figure of 8 with loops of very unequal size.

The author has shown that by depressing the free end so that the non-vertical part slopes downwards and the free end is nearest the earth, the polar curve of radiation can be made more unsymmetrical.

Bellini and Tosi have shown that if a vertical antenna is associated with a nearly closed antenna, and if means are employed to create in both radiators oscillations having a fixed phase difference, then we can give to the polar curve of radiation in different azimuths a large variety of forms.

For if we denote by A the symmetrical radiation in all azimuths of the vertical open antenna, and if we denote by B the value of the maximum radiation of the closed antenna which is in the plane of that closed antenna, then $B \cos \theta$ will denote the radiation at an equal distance in any other azimuth inclined at an angle θ to the plane of the closed antenna. Hence if ϕ is the phase difference of the oscillations in the open and closed antenna, the resultant field in any azimuth at equal distances will be represented by I where I is given by

$$I^2 = (A + B \cos \theta \cos \phi)^2 + (B \cos \theta \sin \phi)^2$$
$$= A^2 + B^2 \cos^2 \theta + 2AB \cos \theta \cos \phi.$$

If then $\phi = 0$, or the phase difference is zero, we have

$$I = A + B \cos \theta.$$

The polar curve of radiation is therefore a curve whose radii vectores for various values of θ are proportional to I.

The curve denoted by $r = A + B \cos\theta$ is called a Limaçon. If $A = B$ the curve is a *cardiod* or heart-shaped curve, and for suitable values of A and B may be made to represent a *figure of* 8 *curve* with unequal loops.

An antenna of more or less closed form, such as a bent or directive antenna, has some of the properties of an open antenna, viz. its radiative power, and some of the properties of a closed antenna, viz. its unsymmetrical radiation. Thus we find that such semi-closed antennæ have smaller radiation decrements than simple linear or open aerials.

7. Connection between the Radiation Resistance and Decrement

There is a close connection between the *radiation resistance* (R_a) of an antenna and its *radiation decrement* δ_r, which is as follows :—

In any closed circuit having a high frequency resistance R' and inductance L, if there be oscillations of frequency L, then the resistance decrement or decrement due to resistance is

$$\delta = \frac{R'}{4NL}. \quad . \quad . \quad . \quad (52)$$

This gives the decrement per semiperiod.

Hence $R' = 4NL\delta$, and if we know or can find by Bjerknes' method the decrement we can calculate the resistance. The same formula applies to the radiation decrement δ_r. For if R_a is the radiation resistance, then

$$R_a = 4NL\delta_r. \quad . \quad . \quad . \quad (53)$$

If R_a is expressed in ohms, or is the factor by which we must multiply the R.M.S. current in amperes in the oscillator to obtain the total radiation in watts, then we must remember that 1 ohm $= 10^9$ electromagnetic units of resistance and write the above formula in the form

$$\delta_r = \frac{R_a \times 10^9}{4NL}. \quad . \quad . \quad . \quad (54)$$

Let C be the capacity of an oscillator, with respect to the earth, reckoned in electrostatic units, then from the dimensions of capacity in electromagnetic units (see Chap. II.) it is seen that to express capacity in E.M. units (C_1) we have to divide capacity expressed in E.S. units by 9×10^{20}.

Hence multiplying the numerator of (54) by C and the denomi-

nator by $C_1 \times 9 \times 10^{20}$, and bearing in mind that L is reckoned in electromagnetic units and that $C_1 L = \frac{1}{4\pi^2 N^2}$ we have

$$\delta_r = \frac{10^9 \pi^2 NC}{9 \times 10^{20}} R_a$$

or, since $N\lambda = 3 \times 10^{10}$, we have

$$\delta_r = \frac{\pi^2 C}{30\lambda} R_a \quad . \quad . \quad . \quad (55)$$

or approximately, since $\pi^2 = 9.87$,

$$\delta_r = \frac{C}{3\lambda} R_a. \quad . \quad . \quad . \quad (56)$$

We can therefore find the radiation decrement from the radiation resistance, and *vice versa*, when we know C and λ for the oscillator.

Thus we take the following values for the radiation resistance as already calculated :—

For a Hertzian dipole oscillator of total length l

$$R_a = 80\pi^2 \frac{l^2}{\lambda^2} \text{ (ohms), } \textit{see } (28). \quad . \quad . \quad (57)$$

For a Flat-top antenna of vertical height h

$$R_a = 160\pi^2 \frac{h^2}{\lambda^2} \text{ (ohms), } \textit{see } (32). \quad . \quad (58)$$

For a plain single wire aerial of height h

$$R_a = 640 \frac{h^2}{\lambda^2} \text{ (ohms), } \textit{see } (33). \quad . \quad . \quad (59)$$

Hence multiplying these expressions by $\frac{\pi^2 C}{30\lambda}$ we have for the radiation decrements :—

For the Hertzian oscillator

$$\delta_r = \frac{8}{3} \frac{\pi^4 l^2 C}{\lambda^3}. \quad . \quad . \quad . \quad (60)$$

For the Flat-top antenna

$$\delta_r = \frac{16}{3} \frac{\pi^4 h^2 C}{\lambda^3}. \quad . \quad . \quad . \quad (61)$$

For the plain aerial wire

$$\delta_r = \frac{64}{3} \frac{\pi^2 h^2 C}{\lambda^3}, \quad . \quad . \quad . \quad (62)$$

where C is the capacity of the antenna in electrostatic units or centimetres.

To obtain the total decrement of the oscillator or antenna we have to add to this the high frequency resistance of the antenna wire and any coil in series with it, and also that of any spark in series with it and that of any earth plate or earth resistance. Denoting all this high frequency resistance in ohms by R' and the total decrement by D we have

$$D = \frac{\pi^2 C}{30\lambda}\left\{R' + R_a\right\}. \quad . \quad . \quad (63)$$

The power radiated from the antenna reckoned in watts is

$$P = R_a a^2$$

where a is the R.M.S. current at the centre or at the base. The *radiation efficiency* (η) of an antenna is the ratio of the power radiated ($R_a a^2$) to the total power given to the antenna which is $(R' + R_a)a^2$.

Hence
$$\eta = \frac{R_a}{R' + R_a} \quad . \quad . \quad . \quad (64)$$

or
$$\eta = \frac{\delta_r}{D}, \quad . \quad . \quad . \quad . \quad (65)$$

where δ_r is the radiation decrement and D is the total decrement of the oscillations.

As an illustration of the use of these formulæ we take the following :—

Example.—A Hertzian oscillator was constructed by placing two metal rods in line with a spark gap of 1 cm. in length between the inner ends. The rods were each 2 metres long and 0.5 cm. diameter, and they had flat circular disks at the outer ends each 1 metre in diameter. The wave-length omitted by the radiator was 1760 cms. in length. Find the radiation resistance and decrement.

Answer.—The total length of the oscillator is 400 cms. $= l$. The capacity of each disk is $\frac{d}{\pi}$ where d = diameter = 100 cms. Hence the capacity C of one-half of the oscillator with regard to the other is $\frac{100}{2\pi} = 15$ E.S. units nearly.

The inductance L of the rod is approximately given by $2l\left(\log_e \frac{4l}{d} - 1\right) = 800\,(8.05 - 1)$. Hence $L = 5640$ cms.

The capacity reckoned in electromagnetic units is $\frac{15}{9 \times 10^{20}}$.

Hence the frequency of the oscillations N is given by

$$N = \frac{3 \times 10^{10}}{2\pi \sqrt{15 \times 5640}} = \frac{3 \times 10^{10}}{6.28 \times 291} = 17 \times 10^6 \text{ nearly},$$

and $$4NL = 4 \times 17 \times 10^6 \times 5640 = 383{,}520 \times 10^6.$$

Therefore we have the radiation resistance R_a given by

$$R_a = 80\pi^2 \left(\frac{400}{1760}\right)^2 = 40 \text{ ohms (nearly)}.$$

The high frequency resistance R' will be chiefly that due to the spark, which may be taken as about 5 ohms.

Hence the resistance decrement

$$\delta = \frac{R'}{4NL} = \frac{5 \times 10^9}{383.52 \times 10^9} = 0.013,$$

and the radiation decrement

$$\delta_r = \frac{R_a}{4NL} = \frac{40 \times 10^9}{383.52 \times 10^9} = 0.104.$$

Therefore the total decrement is 0.117 per semiperiod. We cannot in this case deduce the number of complete oscillations from the formula (7) in Chapter VII. For the original store of energy given to the oscillator is not sufficient to afford more than about two oscillations. This can be proved as follows :—

The energy stored up in the oscillator before discharge is measured by $\frac{1}{2}CV^2$ where C is the capacity of one-half of the oscillator with respect to the other and V is the spark potential. Now for a 1 cm. spark length the spark potential is 30,000 volts, nearly $= 100$ E.S. units, and the capacity of the oscillator is 15 E.S. units. Hence the energy storage is $\frac{1}{2} 15 \times (100)^2 = 75{,}000$ ergs. The rate at which the energy is radiated is $\frac{16}{3} \frac{\pi^4 \phi^2}{\lambda^3}$ ergs per period.

Now $\phi = CVl = 15 \times 100 \times 400 = 600{,}000$, and $\lambda = 1760$ cms., and $\pi^4 = 100$ nearly. Hence the radiation in ergs per period is

$$\frac{1600 \times 36 \times 10^{10}}{3 \times (1760)^3} = \frac{192 \times 10^{12}}{5.4 \times 10^9} = 35{,}000 \text{ ergs}.$$

Accordingly the energy is all dissipated in two complete oscillations.

8. Practical Determination of Antenna Resistance and Radiation

The practical determination of antenna resistance, that is of the quantity which, when multiplied by the mean-square value of the current at the base, will give the total power loss in the antenna, can be achieved in several ways. The great difficulty is to separate out the radiation resistance from the frictional resistance, that is to determine how much of the power given to the antenna is radiated in the form of electric waves and how much is dissipated in heating the antenna or

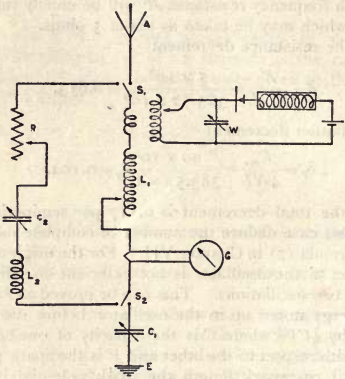


FIG. 30.—Determination of Antenna Total Resistance.

coils in series with it or else in losses in the earth and earth plate.

The total resistance is best determined as follows. Measure the capacity of the antenna and provide an air condenser made with plates of metal parallel and near to each other which has the same capacity. Estimate or measure the inductance of the antenna proper and arrange a switch so that the air condenser can be substituted for the antenna and earth plate, retaining, however, any inductance such as the loading or tuning coil, or the secondary circuit of any jigger or oscillation transformer in series with the antenna (see Fig. 30).

Arrange a variable non-inductive resistance in series with the air condenser and place a thermal ammeter of the kind described in Chapter IV. § 2, to measure the antenna current or the condenser circuit current. Then we have two oscillation circuits, viz. the open or antenna circuit and the closed or condenser circuit, both having equal capacity and inductance, and therefore the same frequency or wave-length. Arrange a coupled exciting circuit with quenched spark discharger to set up oscillations in either of these circuits. Then adjust the variable non-inductive resistance in the condenser circuit so that the hot wire ammeter reads the same current in the two cases. Then this resistance is the total antenna resistance.

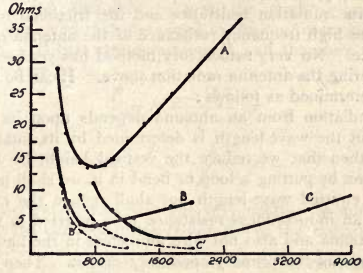


FIG. 31.—Curves showing Variation of Total Antenna Resistance with Wave-length.

If a set of measurements are made with any one antenna by this method and the measured antenna resistance plotted against wave-length for various and increasing wave-lengths, it is found that a curve is obtained which first falls rapidly to a minimum and then rises again.

Theory shows, as we have proved above, that the radiation resistance varies inversely as the square of the wave-length. Hence if the only source of power loss were radiation, the radiation curve should be a curve continually descending as wave-length increases. On the other hand, the curves show that there is also present some source of energy dissipation equivalent to a resistance which increases with the wave-length (see Fig. 31). This is equivalent to a conductivity which increases with the frequency. Now it has been shown by researches made by the author in conjunction with Mr. Dyke that

the conductivity of dielectrics is not constant but increases with frequency (see *Journal of the Institution of Electrical Engineers*, 1912, vol. xlix. p. 323, on "The Power Factor and Conductivity of Dielectrics," by Dr. J. A. Fleming and Mr. G. B. Dyke).

Hence it is probable that this subsequent rise in the antenna resistance curve is due to the dissipation of energy by currents produced in the earth and passing through the earth plate, or induced in the earth around the antenna. The resistance of this earth would then increase with the wave-length up to a certain limit because it involves the conductivity of the dielectric conductors which compose the earth or soil. The great difficulty, however, is to separate out the two resistances, viz. the true radiation resistance and the frictional resistance due to the high frequency resistance of the antenna or of the earth plate. No very satisfactory method has yet been found for measuring the antenna radiation above. It can be approximately determined as follows :—

The radiation from an antenna depends upon its vertical height, but the wave-length is determined by its total length. Suppose then that we reduce the vertical height by lowering the antenna by putting a loop or bend in it, we shall not much alter the emitted wave-length but shall reduce the radiation. Then let an inductionless resistance r be inserted in the base of the antenna and also hot wire ammeters in the base of the antenna and in the primary condenser circuit. Then arrange the value of r, if possible, to comply with the following conditions, viz. to make the root-mean-square current a_2 in the base of the antenna the same and also the condenser circuit current a_1 the same, and also let the antenna give the same wave-length λ, both when of the height h_1 before lowering and when lowered to the height h_2, and the resistance r inserted in the base. Then if R' is the rest of the antenna ohmic resistance, including that of the earth plate, and if K is the radiation constant, we have the following equations :—

(i.) Before lowering the antenna

$$\left(K\frac{h_1^2}{\lambda^2}+R'\right)a_2^2 = \text{Power given to the antenna} = P.$$

(ii.) After lowering the antenna

$$\left(K\frac{h_2^2}{\lambda^2}+R'+r\right)a_2^2 = \text{Power given to the antenna} = P.$$

The condenser current remains the same, and if the antenna

current remains the same then we can equate the two expressions, and we have

$$K\frac{h_1^2}{\lambda^2} + R' = K\frac{h_2^2}{\lambda^2} + R' + r.$$

Hence
$$K\frac{h_1^2 - h_2^2}{\lambda^2} = r$$

or
$$K = \frac{\lambda^2}{h_1^2 - h_2^2} r. \quad . \quad . \quad . \quad (66)$$

This determines the radiation constant.

Hence if we have measured the power actually given to the antenna as the difference between the power given out by the condenser and that dissipated in the condenser circuit (see Chap. IX. § 1), we can determine P, and therefore the value of R', since we have

$$\frac{P}{a_2^2} = R' + \frac{h_1^2}{h_1^2 - h_2^2} r$$

or
$$R' = \frac{P}{a_2^2} - \frac{h_1^2}{h_1^2 - h_2^2} r. \quad . \quad . \quad (67)$$

The method is, however, not very reliable because we are obtaining the two quantities required, viz. the radiation resistance $K\frac{h_1^2}{\lambda^2} = R_a$ and the Joulean or frictional resistance R′ as the difference of two other observed quantities, and any percentage error made in the taking the quantities themselves makes a much larger percentage error in their difference.

Hence estimates of the true power radiated as electric waves from an antenna are likely to be very wide of the mark in the absence of any true radiation meter.

9. The Predetermination of the Natural Frequency of an Antenna and Jigger or Tuning-coil in Series with it

If a radiating or receiving antenna consisting of a wire or wires having distributed capacity and inductance is placed in series with coils of wire such as jigger circuits or tuning-coils, we cannot determine the natural frequency of such a loaded antenna merely by taking the reciprocal of the square root of the product of the capacity and total inductance multiplied by 2π.

Thus if C is the total capacity to earth of such an antenna, and L_1 is the total distributed inductance, and L_2 is the localised inductance of any coil in series with the antenna, then a considerable error would be committed in reckoning the frequency of such an antenna as equal to $\frac{1}{2\pi\sqrt{C(L_1+L_2)}}$. In such cases we have to calculate the frequency by a different method, as shown by Mr. Louis Cohen (see *Electrical World*, January 30, 1915).

Let L and C be the inductance and capacity of the antenna per unit of length, and let the resistance be negligible. Then if x is any distance measured from a fixed point, and if V and I are the potential and current (maximum values) at x, we have the equations

$$L\frac{dI}{dt} = -\frac{dV}{dx}, \quad . \quad . \quad . \quad (68)$$

$$C\frac{dV}{dt} = -\frac{dI}{dx}. \quad . \quad . \quad . \quad (69)$$

By differentiation and substitution we can separate the variables as follows :—

$$LC\frac{d^2I}{dt^2} = \frac{d^2I}{dx^2}, \quad . \quad . \quad . \quad (70)$$

$$LC\frac{d^2V}{dt^2} = \frac{d^2V}{dx^2}. \quad . \quad . \quad . \quad (71)$$

The solution of these equations is

$$I = \left(A_1 \cos pt + A_2 \sin pt\right)\left(B_1 \cos qx + B_2 \sin qx\right), \quad (72)$$

$$V = \sqrt{\frac{L}{C}}\left(A_1 \sin pt - A_2 \cos pt\right)\left(B_1 \sin qx - B_2 \cos qx\right), \quad (73)$$

where

$$p = \frac{q}{\sqrt{LC}}.$$

The frequency of the oscillations is given by

$$n = \frac{p}{2\pi} = \frac{q}{2\pi\sqrt{LC}} = \frac{ql}{2\pi\sqrt{L_1C_1}}, \quad . \quad . \quad (74)$$

where l is the length of the antenna and C_1 and L_1 its tota distributed capacity and inductance.

To obtain the constants in the equations (72) and (73) we proceed as follows :—

Let L_0 be the inductance of the coil in series with the antenna, and let distance x be measured from the earthed end of the antenna. Then when $x = l$ we have $I = 0$, and when $x = 0$ we have $V + L_0 \frac{dI}{dt} = 0$. Substituting these values in (72) and (73) we have

$$B_1 \cos ql + B_2 \sin ql = 0 \quad . \quad . \quad (75)$$

$$B_2 \sqrt{\frac{L}{C}} + B_1 p L_0 = 0. \quad . \quad . \quad (76)$$

Hence eliminating B_2/B_1 from these last equations we have

$$\cot ql = pL_0 \sqrt{\frac{C}{L}} = \frac{qL_0}{L} = ql \frac{L_0}{L_1}. \quad . \quad . \quad (77)$$

Therefore we have

$$\frac{\cot ql}{ql} = \frac{L_0}{L_1}. \quad . \quad . \quad . \quad (78)$$

If we are given the ratio of the inductance of the loading coil to the total inductance of the antenna we can find ql and hence n since we have

$$n = \frac{ql}{2\pi \sqrt{L_1 C_1}}. \quad . \quad . \quad . \quad (79)$$

To solve equation (78) we need a table giving us the values of cot θ/θ for various angles θ reckoned in radians as follows :—

θ.	Cotangent θ.	Cot $\theta/\theta = L_0/L_1$.
0	∞	∞
0.1	9.96	99.6
0.2	4.93	24.65
0.3	3.23	10.77
0.4	2.37	5.92
0.5	1.830	3.65
0.6	1.461	2.435
0.7	1.187	1.696
0.8	0.971	1.214
0.9	0.794	0.882
1.0	0.642	0.642
1.1	0.509	0.463
1.2	0.389	0.324
1.3	0.278	0.214
1.4	0.1724	0.123
1.5	0.0709	0.047
$1.57 = \frac{\pi}{2}$	0.0000	0.000

From the above table we can construct a curve giving us by interpolation the value of ql corresponding to any given value of L_0/L_1 as follows:—

L_0/L_1.	ql.
0.0	$1.57 = \pi/2$
0.05	1.51
0.10	1.426
0.20	1.314
0.30	1.219
0.40	1.142
0.50	1.078
0.60	1.022
0.70	0.968
0.80	0.925
0.90	0.892
1.00	0.855
1.50	0.735
2.00	0.650
2.50	0.59
3.00	0.545
3.50	0.510
4.00	0.475
5.00	0.430
6.00	0.40
7.00	0.370
8.00	0.350
9.00	0.325
10.00	0.31
25.00	0.19

If, then, we are given the ratio L_0/L_1, viz. the ratio of the inductance of the loading coil to the total inductance of the whole antenna, all we have to do is to look out this ratio in the first column of the last table, and opposite will be found the proper value of ql. Then the frequency of that antenna is given by

$$n = \frac{ql}{2\pi\sqrt{L_1 C_1}},$$

and the radiated wave-length λ reckoned in kilometres is obtained by dividing 300,000 by this calculated value of n.

Example.—An antenna has a total capacity of 0.001 microfarad and a total inductance of 0.05 millihenry. In series with it is placed a tuning-coil having an inductance of 0.05 millihenry. Find the natural frequency of this loaded antenna and the wave-length of the radiated waves.

Answer.—We have $L_0 = 0.05$ and $L_1 = 0.05$. Therefore $L_0/L_1 = 1$ and $C_1 = 0.001$ mfd.

Looking out in the last table the value of $L_0/L_1 = 1$ we

find opposite to it $ql = 0.855$. Hence the true natural frequency n is

$$n = \frac{0.855}{6.283\sqrt{5 \times 10^{-5} \times 10^{-9}}} = 610{,}700.$$

If we had determined the frequency by the incorrect formula

$$n = \frac{1}{2\pi\sqrt{C_1(L_0 + L_1)}}$$

we should have obtained $n = 503{,}200$, or an error of 16 per cent.

It will be seen, therefore, that this correction is an important one. If the true frequency is obtained experimentally by the cymometer we can employ this observed value of n to give us the true inductance of the antenna when we know its total capacity C_1 and the inductance L_0 of the loading coil.

CHAPTER IX

TRANSMITTERS AND TRANSMITTING CIRCUITS

1. Spark Transmitters

By far the largest portion of all practical wireless telegraphy is conducted by means of spark transmitters. In this arrangement the oscillations are produced in the antenna by means of free condenser discharges which are made to take place with great regularity and at exactly equal intervals.

The transmitter comprises therefore :—

1. A condenser which may consist of Leyden jars or of metal plates in oil, with or without glass plates between them, or of metal plates hung up in air.

2. Some means of charging the condenser, which may be a battery of high E.M.F., an induction coil, or an alternating current transformer and alternater.

3. A discharger for discharging the condenser at regular intervals through a circuit and across a spark gap.

4. An oscillation transformer or jigger, by means of which the discharges of the condenser are made to produce trains of feebly damped oscillations in the antenna.

We have then two frequencies to consider :—

1. The *oscillation frequency*, the reciprocal of which is the time-period of the oscillations.

2. The *spark frequency*, or number of spark discharges per second.

The former can be measured by the cymometer, as already explained (see Chap. V.), and the latter by means of the Fleming Photographic Spark Counter, which records on a moving photographic plate an image for each spark, and from the speed of the plate and from the number of images in a given space we can determine the number of sparks per second.

In modern spark transmitters the spark discharger is of

such a type as to give as far as possible a quenched or dead beat discharge of the condenser. This means that the condenser discharge consists of very few oscillations and the condenser circuit becomes open almost immediately the spark starts. This is best achieved by Marconi's high speed rotating dischargers, in which a rapidly rotating disk, generally fixed on the shaft of the alternater, has a number of spokes set in its edge so that it resembles a rimless wheel. There are two fixed spark balls or surfaces so placed that the spokes just graze against them when rotating, and these contacts take place simultaneously with two spokes, so that for a single instant a circuit is closed through which the condenser discharge takes place. It is then opened again directly and the draught of air produced by the rapidly rotating disk blows out any arc discharge which is formed.

If this nearly dead beat discharge of the condenser passes through one coil of an oscillation transformer, the other coil of which is in series with the antenna, and if the antenna circuit is tuned to the condenser circuit, then we have set up in the antenna long trains of feebly damped oscillations which have the frequency of the free antenna. If the coupling between the two circuits is weak, that is, not more than 10 or at most 15 per cent, then the frequency of the waves radiated from the antenna will have a single frequency.

To study the efficiency and performance of such a transmitter we require to make the following measurements, which can all be made by the appliances and methods already described.

1. The *electrostatic capacity* (C) of the antenna. This is best measured by the Fleming and Clinton Commutator method (see Chap. V. § 3). It is best to make sure first that the insulation of the antenna is good. This is most easily accomplished by the aid of any of the "Megger" or other testing sets used for measuring the insulation of electric-light house-wiring. The insulation chiefly depends on the surface insulation of the insulators upholding the antenna at the top, and this is greatest in dry frosty weather and worst in wet weather. It is better to make the insulation test over several days in various weathers, but to select a dry day for the capacity test. The capacity can be expressed in microfarads or in micromicrofarads by multiplying by 10^6, or in centimetres or electrostatic units by multiplying the capacity in microfarads by 9×10^5.

2. The *earth-plate resistance* can only be determined if the

earth plate has been put down in three separate portions or plates at a little distance from each other. If it has been so made then we can approximately determine the earth resistance by successively placing a single storage cell having an E.M.F. of 2 volts in series with an ammeter and two of earth plates taken pair and pair, and determining the current which flows through the circuit when the outer terminals of the battery + ammeter are connected to the two earth plates. The direction of the E.M.F. should be rapidly and frequently reversed by a switch, and the ammeter should be a low-resistance hot-wire ammeter, the deflections of which are independent of the direction of the current. This test will merely give an approximation to the resistance, but it should indicate not more than 1 or 2 ohms; that is to say, a cell of 2 volts E.M.F. should give a current of 1 to 2 amperes through the earth when used in the above manner. We can then calculate the joint resistance of the triple earth plate as follows:—

Let r_1, r_2, r_3 be the resistances to earth from each plate or part separately. Then when we measure with a voltage V the current A, which flows from the plate 1 to the plate 2, we have the equation

$$\frac{V}{A_1} = R_1 = r_1 + r_2. \quad . \quad . \quad . \quad (1)$$

Similarly for plates 2 and 3 we have

$$\frac{V}{A_2} = R_2 = r_2 + r_3, \quad . \quad . \quad . \quad (2)$$

and also for plates 1 and 3 we have

$$\frac{V}{A_3} = R_3 = r_1 + r_3. \quad . \quad . \quad . \quad (3)$$

Hence subtracting (1) and (2) we have

$$R_1 - R_2 = r_1 - r_3$$

and

$$R_3 = r_1 + r_3.$$

Hence

$$R_1 - R_2 + R_3 = 2r_1 \quad . \quad . \quad . \quad (4)$$

and

$$R_3 - R_1 + R_2 = 2r_3, \quad . \quad . \quad . \quad (5)$$

and from equations (2) and (5) we have

$$R_1 + R_2 - R_3 = 2r_2.$$

Therefore we have

$$r_1 = \frac{V}{2}\left(\frac{1}{A_1} - \frac{1}{A_2} + \frac{1}{A_3}\right), \quad . \quad . \quad (6)$$

$$r_2 = \frac{V}{2}\left(\frac{1}{A_1} + \frac{1}{A_2} - \frac{1}{A_3}\right), \quad . \quad . \quad (7)$$

$$r_3 = \frac{V}{2}\left(-\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3}\right), \quad . \quad . \quad (8)$$

and therefore the total or joint resistance R of all three plates in parallel is

$$R = \frac{1}{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}}. \quad . \quad . \quad . \quad (9)$$

Example.—With a single storage cell the current flowing when it was connected to plates 1 and 2 of a triple earth plate was 1 ampere. When connected to plates 2 and 3 it was 2 amperes, and to plates 1 and 3 it was 1.5 amperes. Find the total or joint resistance of all three plates in parallel.

Answer.—The voltage $V = 2$ volts, hence we have

$$r_1 = (1 - \tfrac{1}{2} + \tfrac{2}{3}) = \tfrac{7}{6} \text{ ohm},$$
$$r_2 = (1 + \tfrac{1}{2} - \tfrac{2}{3}) = \tfrac{5}{6} \text{ ohm},$$
$$r_3 = (-1 + \tfrac{1}{2} + \tfrac{2}{3}) = \tfrac{1}{6} \text{ ohm}.$$

Then the joint resistance R is

$$R = \frac{1}{\frac{6}{7} + \frac{6}{5} + 6} = \frac{35}{282} \text{ ohm},$$

or $\frac{1}{8}$th of an ohm nearly.

All earth plates and lightning earth plates should be put down in three parts not too near together, and the three parts tested separately but used in parallel.

Measurements of earth-plate resistance made with a battery and ammeter in the above manner do not, however, give us the true earth-plate resistance for high frequency oscillations. They only give a rough indication whether the "earth" is good or bad.

The resistance of conducting and electrolysable dielectrics such as clay, rock, sand, moist earth or the usual foundation materials have much greater conductivity for high frequency currents than for continuous currents, and they have a maximum conductivity for a certain high frequency.

Hence the "earth resistance" we are concerned with in

wireless telegraphy is a much smaller quantity than would be indicated by the tests made as described with an ammeter and battery cell and triple earth plate. Nevertheless it is useful to make the test.

3. The *capacity* of the main *condenser* should then be determined by the Fleming and Clinton Commutator, and if possible the equivalent resistance of these condensers for various currents through them, so that the energy loss can be calculated. This necessitates an air condenser of equal capacity to one single section of this condenser. Suppose, for instance, that the condenser consists of a large number of glass tubular condensers, we can determine the equivalent resistance for each one as described in Chapter VII. § 4. If the condenser consists of metal plates in oil, with or without glass plates between, we can in the same manner test each separate section. It is desirable to use metal plates in paraffin oil carefully freed from water if small condenser losses are to be attained.

4. The cymometer can then be employed to determine a *resonance curve* and the *decrement* of the antenna circuit and the condenser circuit, as already described. Knowing the frequency N of the oscillations and the capacities both of the antenna circuit and of the condenser circuit, we can determine the *inductances* L_2 and L_1 of these two circuits, for we have the equations

$$2\pi N = \frac{\sqrt{10^{15}}}{\sqrt{C_1 L_1}} = \frac{\sqrt{10^{15}}}{\sqrt{C_2 L_2}},$$

where C_1 and C_2 are capacities reckoned in *microfarads*, and the inductances L_1 and L_2 are reckoned in *centimetres*, and N is the oscillation frequency. Hence, knowing N, C_1 and C_2, we can find L_1 and L_2. Also from the resonance curves we have the decrements per semiperiod D_1 and D_2 of the two circuits. We can then find the high frequency resistance R_1 and antenna total resistance R_2, for we have the equations

$$\left.\begin{aligned} 4NL_1D_1 &= R_1 10^9 \\ 4NL_2D_2 &= R_2 10^9 \end{aligned}\right\}, \qquad . \quad . \quad (10)$$

where R_1 and R_2 are reckoned in ohms and L_1 and L_2 in centimetres.

But $$R_2 = R_a + R^1$$

where R_a is the antenna radiation resistance and R^1 is the frictional resistance whether due to the wire or to the earth resistance.

We can find the total resistance of the antenna by the method described at the end of Chapter VIII., viz. by substituting for the antenna a closed condenser circuit in which is placed an air condenser equal in capacity to the antenna in series with the same inductance as exists in the real antenna. We have then to add also in series a variable non-inductive resistance, the high frequency resistance of which can be determined. Also a hot wire ammeter is included in the circuit. The capacity and resistance are then separately adjusted, so that the closed condenser circuit has the same time-period of oscillation as the antenna circuit and the same mean-square current in it. We then know that the resistance inserted in the closed condenser circuit is the same as the total resistance of the antenna.

This measurement should be made for several different frequencies. This total antenna resistance should not exceed 5 or 10 ohms at the wave-length at which the antenna is to be worked, and should preferably be less. If it is more than 10 ohms it indicates an imperfect or bad earth connection.

5. We then employ suitable hot wire ammeters, constructed as described in Chapter IV. § 2, to determine the antenna current and the current in the condenser circuit. Let these currents reckoned in amperes be denoted by A_2 and A_1.

The power wasted as heat in the condenser circuit and in the spark is then equal to $R_1A_1^2$ watts and that dissipated in the antenna and earth as heat together with that radiated as electric waves is given by $R_2A_2^2$.

The total rate of dissipation of energy is then

$$R_1A_1^2 + R_2A_2^2.$$

6. The *spark voltage* V_1 can then be determined from the spark length and the spark voltage tables, and by the aid of a Fleming Photographic Spark Counter the number of discharges per second can be found or the spark frequency N^1.

If then C_1 is the condenser capacity and V_1 the spark voltage the energy stored in the condenser before each discharge is $\frac{1}{2}C_1V_1^2$, and the rate at which energy is given to the primary oscillation circuit is $\frac{1}{2}N^1C_1V_1^2$.

This power is employed in the generation of heat in the primary circuit and in the antenna earth plate and in the wave radiation. Hence if there is no other source of loss we should have the equation

$$\frac{1}{2}N^1C_1V_1^2 = R_1A_1^2 + R_2A_2^2. \quad . \quad . \quad (11)$$

Since all the quantities in the above equation can be measured, we can test the equation.

7. If the condenser is charged by an alternating current transformer we can by means of a properly constructed watt-meter measure the power being given to it on the low tension side. This power will be partly dissipated in the coils and core of the transformer and partly in the choking coils which are interposed between the transformer and the condenser, and the remainder will be transferred to the condenser.

The best method of representing and setting out all these losses is by an energy balance-sheet as first done by the author in a paper on "Quantitative Measurements in Radio-telegraphy" (see *Journal Institution Electrical Engineers*, vol. xliv. p. 344, 1910).

It was applied as shown in the figures below to a small wireless transmitter at University College, London, which is purely an experimental plant and not therefore to be taken as illustrative of the losses in actual transmitters, but merely to show the method.

The appliances, such as the rotary transformer, alternating current transformer, were those in use for other purposes in the laboratory and not made especially for radiotelegraphy. The plant takes continuous current from the public electric supply circuits at 210 volts, and transforms it first by a rotary generator into alternating current having a frequency of 100.

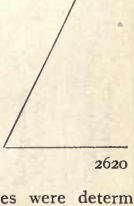
This low voltage A.C. is then raised by a static transformer to 20,000 volts pressure and used to charge a glass plate condenser. This condenser is in series with a stationary spark gap supplied with an air blast and with one coil of an oscillation transformer, the other coil being in series with the antenna.

Certain choking coils are introduced between the rotary and static transformer, and between the static transformer and condenser, which absorb power, but which are necessary to prevent oscillations travelling back into the transformer coils and breaking down the insulation.

The power given to the rotary transformer in the form of direct current was 2620 watts, and this is an item to the credit of the apparatus. The debtor side contains all the losses as follows:—

POWER BALANCE-SHEET FOR AN EXPERIMENTAL RADIO-TELEGRAPHIC TRANSMITTER

Cr.	Watts.	*Dr.*	Watts.
By Power given to rotary transformer in form of direct current, 12 amperes (nearly) at 220 volts.	2620	To Rotary losses . .	1139
		,, Line losses . . .	71
		,, Choker losses . .	610
		,, Transformer losses .	355
		,, Oscillation circuit losses in condenser, spark gap, and inductance .	351
		,, Power given to the antenna, partly dissipated by resistance and partly in the form of electromagnetic waves together with any unknown losses in earth plate . .	94
	2620		2620



These losses were determined by measurement and calculation from the measured constants of the transmitter as follows :—

N_1 = Spark frequency = 208.
S = Spark length = 5 mm.
C_1 = Condenser capacity = 0.0139 mfd.
V_1 = Spark voltage = 17,500 volts.
O_1 = Oscillation constant of condenser circuit = 6.9.
A_1 = Current in condenser circuit R.M.S. value = 26.5 amps.
λ = Wave-length of radiated waves = 1350 feet.
D_1 = Semiperiod decrement of condenser circuit = 0.0507.
A_2 = Antenna current R.M.S. value = 4.9 amps.
R_2 = Total resistance of antenna circuit made up of 0.528 ohm for jigger secondary, 0.21 ohm for antenna proper, and in addition earth resistance and antenna radiation resistance = 3.9 ohms.

The antenna consisted of four copper wires, each 2/16, about 60 feet vertical and 90 feet horizontal, forming an inverted L antenna. The length of the jigger wire (7/22) was 45 feet, the total length of antenna being 135 feet. The four antenna wires were spaced 8 to 10 feet apart.

h = The vertical height of the antenna = 60 feet.

R_a = The radiation resistance of the antenna $= 1600\frac{h^2}{\lambda^2}$

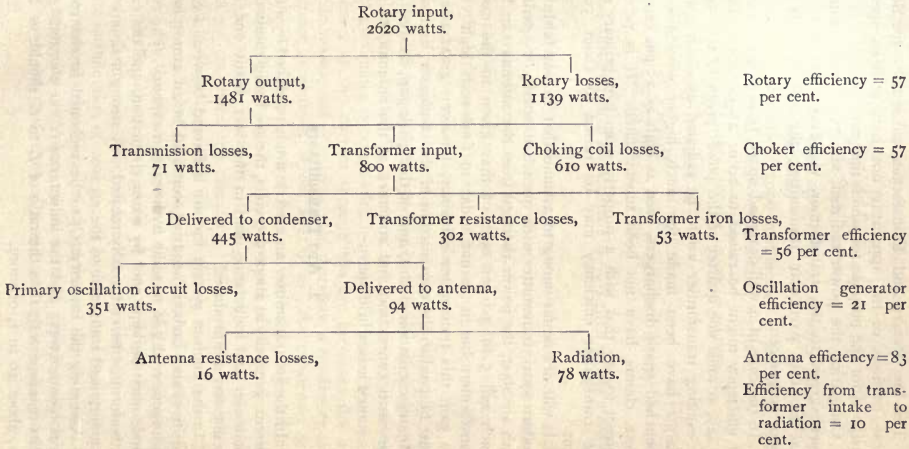
$= 1600\left(\frac{60}{1350}\right)^2 = 3.15$ ohms.

P = Power radiated as electric waves $= R_a A_2^2 = 75.6$ watts.

Since the radiated power calculated from the calculated antenna radiation resistance is almost as great as the measured power taken up by the antenna, obtained as a difference of the credit and debtor energy account, it follows that there is some inaccuracy, as no sufficient margin exists for the earth-plate losses. We can, however, set out the losses and efficiencies of the various parts of the transmitter approximately as follows :—

[TABLE

EFFICIENCY CHART FOR U.C.L. RADIOTELEGRAPHIC EXPERIMENTAL PLANT, WITH 5 MM. SPARK GAP



The low Rotary and Transformer efficiencies are due to these portions of the apparatus not having been specially designed for the purpose for which they are used.

Comparing the power radiated, viz. 75-78 watts, with the power given to the transformer, viz. 800 watts, it is seen that for this particular transmitter the efficiency is only about 10 per cent.

In the case of a well-designed spark transmitter the overall efficiency might be raised to 20 or 25 per cent; that is to say, at most one quarter of the power given to the step-up or exciting transformer would be radiated as electric waves. Claims have been made for much higher efficiencies for quenched spark dischargers, even as high as 75 per cent, but these claims have not been substantiated. It is probably correct to say that with a properly quenched primary spark the transmitter efficiency may be raised to about 50 per cent.

It will not be far wrong to assume that in any fairly good modern spark transmitter with quenched spark, or spark gap with air blast, the power radiated from the antenna can be as much as 25 to 30 per cent of that given to the exciting transformer, but it is not usual to find such high values. To gain this transmitter efficiency, however, great attention must be paid to the construction of the condenser, and of the primary oscillation circuit, and of the spark gap, as well as to the earth connection and structure of the antenna, to avoid resistance losses in them.

2. Arc Transmitters

In the arc transmitter an electric arc is produced by a direct current, generally at about 500 volts, the arc being shunted by a condenser and inductance, and in this last circuit undamped oscillations are produced.

In the Poulsen system the arc is formed between a copper (positive) and carbon (negative) terminal in an atmosphere of hydrogen, coal gas, or some hydrocarbon vapour. The copper anode must be kept cool by water circulation, and the carbon cathode must be continually rotated. A strong magnetic field is maintained transverse to the arc.

Assuming the capacity (C) to be reckoned in electrostatic units (centimetres), and the inductance (L) in electromagnetic units (centimetres), then the ratio of L to C should be large, viz. about 20 to 1, or more.

These oscillations then have a frequency

$$N = \frac{3 \times 10^{10}}{2\pi \sqrt{C_{ES} L_{EM}}},$$

where C_{ES} stands for the capacity in centimetres and L_{EM} for the inductance in centimetres.

Thus if a capacity of 2000 centimetres is joined in series with an inductance of 50,000 centimetres, we have

$$\sqrt{C_{ES} L_{EM}} = \sqrt{2000 \times 50{,}000} = 10{,}000,$$

and the frequency N would be

$$N = \frac{3 \times 10^{10}}{6.28 \times 10{,}000} = \frac{.10^6}{2.09} = 500{,}000 \text{ nearly.}$$

The time-period of oscillation $T = \frac{1}{N}$ is then nearly 2 *micro-seconds*.

The explanation of the arc generator is to be found in the fact that the *characteristic curve* of the arc is a downward sloping curve with concavity upwards.

The characteristic curve of any power-absorbing device is a curve which delineates the relation between the current flowing into it and the terminal potential difference.

Thus, if we set off current in amperes horizontally, and potential difference of the electrodes in volts vertically, the characteristic of the direct current arc in air is a curve such that the ordinates decrease with increase of the abscissæ.

Hence, as the current through the arc increases the arc P.D. decreases, and *vice versa*. If, therefore, a condenser and inductance is shunted across a D.C. arc, the condenser first robs the arc of some current, because current is deflected into the condenser to charge it. Thus decrease of current increases the P.D. at the terminals of the condenser, and therefore assists or continues the charging. When the condenser is charged it discharges back through the arc and increases the arc current, and thus decreases the arc P.D. and promotes discharge. So the current flows into and out of the condenser in time with the natural time-period of the condenser-inductance circuit, and thus establishes undamped oscillations in it.

The introduction of coal gas or hydrocarbon vapour round the arc steepens the characteristic curve, and therefore enables more energy to be taken up in a small capacity, which is essential for the production of energetic high frequency oscillations.

It is important to notice that the potential difference (P.D.) at the terminals of the condenser, as measured by an electrostatic voltmeter, may be many times greater than the arc P.D. as measured by a direct current voltmeter.

Let V_0 be the P.D. of the carbon and copper electrodes as measured by a direct current voltmeter, and let V be the R.M.S. value of the P.D. at the terminals of the condenser as measured by an electrostatic voltmeter. This last reading is then the root-mean-square value of the sum or difference of the direct or arc P.D. and the alternating P.D. produced by the ebb and flow of the high frequency current through the condenser. Suppose the capacity of the condenser is C in electrostatic units, and let A be the R.M.S. value of the alternating high frequency current flowing through the condenser with frequency $N = \frac{p}{2\pi}$. Then the condenser terminal P.D. V' produced by the alternating current is given by

$$A = \frac{CpV'}{9 \times 10^{11}},$$

or

$$V' = \frac{9 \times 10^{11} A}{2\pi NC}.$$

This is the R.M.S. value. Hence the maximum value is $\sqrt{2}V'$, and the instantaneous value at any time t is

$$\sqrt{2}V' \sin pt.$$

Accordingly the actual measured value of the P.D. at the terminals of the condenser is given by

$$V = \sqrt{\frac{1}{T}\int_0^T \left(V_0 \pm \sqrt{2}V \sin pt\right)^2 dt},$$

$$= \sqrt{V_0^2 + V'^2}.$$

Hence V may be very much greater than V_0.

Example.—Suppose the condenser shunted across an arc has a capacity of 4000 cms., and the inductance in series with it measures 100,000 cms.

Let the direct P.D. across the arc be 400 volts and the current through the condenser circuit be 10 amperes, what is the frequency of the undamped oscillations set up and the P.D. across the terminals of the condenser measured with an electrostatic voltmeter?

Answer.—The frequency

$$N = \frac{3 \times 10^{10}}{2\pi\sqrt{4000 \times 100{,}000}}$$

$$= \frac{3}{4\pi} \times 10^6 = 0.24 \times 10^6 \text{ nearly.}$$

Hence $$2\pi N = p = 1.5 \times 10^6.$$

Also if V' is the R.M.S. value of the potential difference at the terminals of the condenser due only to the high frequency current of 10 amperes, we have

$$10 = \frac{4000 \times 1.5 \times 10^6 \times V'}{9 \times 10^5 \times 10^6},$$

or $$V' = 1500.$$

Hence, since $V_0 = 400$, we have

$$V = \sqrt{(400)^2 + (1500)^2} = 1552 \text{ volts.}$$

Therefore the frequency is 240,000 and the electrostatic voltmeter would read 1552 volts.

3. High Frequency Alternators

In ordinary spark stations alternators are employed which give an alternating current having a frequency from 300 to 500 or even 1000, but these are not, properly speaking, high frequency (H.F.) alternators. That term is now restricted to alternators giving a current having a frequency from 40,000 to 100,000 or more which can be used to excite directly persistent oscillations in the antenna without the interposition of condenser discharges.

These H.F. alternators are of two kinds: (1) ordinary alternators in which the necessary high frequency is obtained by a sufficiently high speed of rotation or by a sufficient number of poles.

This type of alternator may be either a rotating field or armature machine, or it may be an inductor machine in which the revolving part is merely an iron disk with teeth cut on the edge.

If an alternator has P pairs of poles, viz. P north poles and P south poles, on its field magnet, and if the speed of rotation of the field or armature is M revolutions per minute (R.P.M.),

then the speed is $\frac{M}{60}$ revolutions per second (R.P.S.). Hence in the alternating current produced there will be $\frac{PM}{60}$ periods per second, or the frequency N of the alternating current will be

$$N = \frac{PM}{60}.$$

Thus if the machine has 60 poles, or 30 pairs, and a speed of rotation of 1800 R.P.M., then

$$N = \frac{30 \times 1800}{60} = 900.$$

Since for every field pole there must be a corresponding armature coil, and since this necessitates a certain perimeter in the revolving portion, it is easily seen that there is a corresponding peripheral velocity. For if d is the diameter of the revolving part then πd is the perimeter, and since $\frac{M}{60}$ is the time of one revolution in seconds, the linear velocity of a point on the periphery is

$$v = \frac{\pi dM}{60}.$$

Thus if the revolving part has a diameter of 2 feet and a speed of 1800 R.P.M. the peripheral velocity is

$$v = \frac{3.1415 \times 2 \times 1800}{60} = 188.49 \text{ feet per second.}$$

Now, owing to the centrifugal force, there is a limit to the peripheral speed which can be permitted without danger, and this is much lower for such constructions as copper coils wound on iron cores than for solid iron disks or cylinders.

For armatures with coils wound on them in slots this may be taken at 200 to 300 feet per second, and for disks or cylinders of laminated iron at 300 to 400 feet per second. It is therefore seen that there are certain serious limitations to the frequency obtainable by alternators constructed with moving coils or fields. Thus, suppose we take the case of an armature 2 feet in diameter or 6.28 feet in circumference and allow 1 inch per pair of poles or pair of coils. This means at most 75 pairs. Such an armature could not be run safely at much more than 3000 R.P.M. or 50 R.P.S. Hence the frequency obtainable would be only $50 \times 75 = 3750$.

By employing steam turbines as motive power and a Mordey type of alternator in which magnetic pole to pole leakage is reduced, Fessenden constructed small H.F. alternators. By the employment of an inductor type of alternator having a disk with many hundred teeth driven at a very high speed by an electric motor, Alexanderson has constructed alternators having a frequency of 100,000 or more.

A more practical solution of the H.F. alternator problem has been given by Dr. R. Goldschmidt in his frequency raising alternator, which is constructed on the following principles. Imagine a single phase alternator with a fixed field or stator and a rotating armature or rotor, and let the stator be excited by a low frequency alternating current of frequency n.

This creates a pulsating or periodic magnetic field fixed in direction but varying in strength according to a simple harmonic law.

It is well known that a magnetic field of fixed direction varying according to a simple harmonic law, say $h = H \sin pt$, can be considered to be composed of two magnetic fields of constant strength $\frac{H}{2}$ revolving in opposite directions with an angular velocity $p = 2\pi n$, where n is the frequency of the periodicity. Thus, if OA and OB represent the oppositely rotating constant magnetic fields of strength $\frac{H}{2}$, their resultant is a periodic field of constant direction but fluctuating strength $H \sin \theta$, where 2θ is the angle between OA and OB at any instant.

Hence the alternating current of frequency n in the stator gives rise to two oppositely rotating fields with angular velocities p. If, then, we make the rotor rotate with an angular velocity p, its coils will cut through one of these fields with an angular velocity $2p$ and be stationary with respect to the other rotating field. This will produce in the rotor an alternating current of frequency $2n$. This current produces a field fixed relatively to the armature which may be resolved into two oppositely rotating fields moving with angular velocities $2n$ with respect to the rotor, but with velocities $2n + n = 3n$ and $2n - n = n$ with respect to the stator.

In order that the current of frequency $2n$ induced in the rotor may circulate, it is necessary to close the circuit of the rotor outside the machine by a circuit which is made up of a condenser in series with an inductance, the capacity being so

chosen that the rotor circuit is tuned for a frequency $2n$. In other words, the capacity C_2 must be such that in conjunction with the whole inductance L_2 inside and outside the rotor we have

$$2\pi \cdot 2n = \frac{10^6}{\sqrt{C_2 L_2}},$$

where C_2 is in microfarads and L_2 in microhenrys.

For this frequency this tuned rotor circuit is non-inductive.

It has been shown in Chapter IV. that when a circuit of resistance R, inductance L, and capacity C in series has an alternating E.M.F. E, and frequency $n = \frac{p}{2\pi}$ applied to it, the current I is given by

$$I = \frac{E}{\sqrt{R^2 + \left(Lp - \frac{1}{Cp}\right)^2}}.$$

Again, if that circuit is tuned for this frequency, we have

$$p = \frac{1}{\sqrt{CL}}, \text{ or } p^2 = \frac{1}{CL}, \text{ or } Lp = \frac{1}{Cp},$$

and hence the current equation becomes $I = \frac{E}{R}$, or the circuit non-inductive for that frequency, and the current and E.M.F. are in step as regards phase.

Accordingly it will be seen that the alternating current of frequency n in the stator induces a current of frequency $2n$ in the rotor, which is 90° different in phase from it because the rotor is made practically non-inductive for currents of the frequency $2n$. Now the rotating fields of angular velocity $3n$ and n produced by this rotor current cut through the stator and induce in it currents of corresponding frequency. But it is clear that the current of frequency n induced in the stator is nearly opposite in phase to the original exciting current of frequency n in the stator. Hence they neutralise each other nearly, and we are left with a current of frequency $3n$ in the stator.

To provide a path for this current the stator is short-circuited by a condenser and inductance so adjusted that in conjunction with the stator inductance the circuit is tuned for a frequency $3n$.

Now this last current in the stator produces two oppositely revolving fields of angular velocity $2\pi \cdot 3n$ or $3 \cdot 2\pi n = 3p$, which

cut through the stator coils with angular velocities relatively to it of $3n+n=4n$ and $3n-n=2n$. But the rotor is already short-circuited for currents of $2n$ by the above-mentioned condenser, and this last current is 180° different in phase from the first-mentioned currents of frequency $2n$ induced in the rotor by the original exciting current. Hence the reaction of the rotating rotor on the stator results in transforming the original exciting current of frequency n in the stator into a stator current of frequency $3n$, whilst we have created in the rotor a current of frequency $4n$.

The current of frequency $4n$ can be transferred to an antenna properly tuned by connecting the antenna to one slip ring of the rotor coils and the earth to the other. This is equivalent to connecting a condenser circuit tuned to a frequency of $4n$ across the rotor circuit. We need not, however, stop at this point. By the use of a closed condenser circuit tuned to a frequency $4n$ across the rotor circuit, we can generate currents of frequency $5n$ in the stator and $6n$ in the rotor and so on, multiplying up to any required extent.

In practice it is best to begin by exciting the stator with an alternating current of fairly high frequency, say 10,000 generated by a separate alternator excited by a direct current generator, we can then by three transformations produce a current of a frequency of 60,000 in the rotor which is high enough for wireless telephony and for long-distance wireless.

It will be seen that the currents of the same frequency occur in pairs in the rotor and in the stator, and that these pairs differ in phase by 180°, and hence nearly annul one another. If it were not for this the machine would have a low efficiency. As a matter of fact, such a machine can have as high an efficiency as any ordinary polyphase or single-phase alternator, provided certain precautions are taken in the construction of the coils on the rotor and of the external condenser circuits required to receive the non-used intermediate harmonics.

The efficiency of such a high frequency alternator can be best measured by receiving the final currents in a condenser circuit, which is all included in a calorimeter, so as to measure the total heat produced in gramme-Centigrade calories or pound-Fahrenheit units.

One gramme calorie is the amount of heat required to raise one gramme of water one degree Centigrade in the neighbourhood of 4° C. It is equal to 42 million ergs or 4.2 joules.

One pound of water raised one degree Fahrenheit in the

neighbourhood of 60° F. represents an energy expenditure of 780 foot-pounds nearly, in the latitude of London.

The best form of calorimeter is a vacuum-jacketed vessel such as a good Thermos bottle used for keeping liquids hot or cold. If such a bottle is filled with paraffin oil and a coil of wire of known resistance R immersed in it, we can pass through the coil a measured direct or continuous electric current A, and then the power given to the coil is A^2R watts, if A is in amperes and R in ohms. If we observe the rise in temperature of the paraffin at regular intervals of time, say one minute, we can plot a curve representing temperature of the oil at equal intervals of time, corresponding, therefore, to equal increments of energy (in joules) given to the coil. If this coil of wire is constructed of fine stranded wire in such a way that its resistance for high frequency alternating currents is the same as for direct currents, we can then use this coil and calorimeter to measure high frequency power. For if we pass the H.F. current for a given time through the coil and observe a certain rise in temperature, we know from the previously constructed curve how much energy (in joules) has been given to the coil to create this rise of temperature.

The condition, however, which must be complied with to secure accuracy is that the observed rise of temperature shall have been attained in about the same number of minutes in the two cases. Hence we must take a number of temperature curves with continuous or direct currents of different ampere values, so that when we make an experiment with an alternating current we can find a curve from this chart which tells us the corresponding continuous current which would create the same rise of temperature in the same time as the H.F. alternating current. We then know the ampere value of this last current, and therefore the power conveyed by it to the coil of known resistance.

CHAPTER X

RECEIVING CIRCUITS AND DETECTORS

1. Receiving Circuits

THE receiving circuits in radiotelegraphy comprise (1) a receiving antenna and tuning coil. (2) A closed receiving circuit containing inductance and capacity which are variable, this being coupled to the antenna. (3) A detector in connection with the condenser circuit, either as a shunt across the condenser or else in series with it. (4) A signal-making instrument which may be a telephone, Einthoven galvanometer, or relay of some kind actuating a Morse Inker or Syphon recorder.

The antenna is the organ by which the incident waves are absorbed and electric oscillations are thereby set up in it. The detector is an appliance which reveals the presence of these feeble oscillations in the receiving antenna. It is therefore necessary that the mean-square value of these oscillations should reach a certain minimum.

For modern telephonic detectors we can say that a current of about 40 microamperes in the antenna is necessary for fairly good signals. The antenna radiates whilst it receives, and when exposed to electric waves falling upon it resembles a body being heated by radiant heat. The temperature rises until the heat lost by radiation, convection, conduction, or in other ways balances the heat received. So in the case of the antenna the current in it when exposed to electric radiation rises until it reaches a value at which the radiated or dissipated energy is equal to the received energy. It is customary to fix upon a certain receiving antenna resistance, say 25 ohms, and to define the energy absorbed as the product of this resistance and the mean-square current at the base of the antenna. Thus, if the received current is 40 microamperes, and if we take the antenna resistance including the oscillation transformer and

tuning coil to be 25 ohms, then the power given to the antenna is $25 \times \left(\frac{40}{10^6}\right)^2 = \frac{4}{10^8}$ watt or $\frac{1}{400}$th of a watt.

This shows that an extremely small amount of energy picked up will create an audible signal.

The relation of the received current to the sending current can be approximately obtained by the use of Dr. W. L. Austin's empirical formula as follows:—

$$I_R = I_S \cdot \frac{K h_1 h_2}{\lambda d} \cdot \epsilon^{-\frac{\alpha d}{\sqrt{\lambda}}} \quad . \quad . \quad (1)$$

where I_R is the receiving antenna current, and I_S the sending antenna current expressed in amperes, h_1 and h_2 are the heights of the sending and receiving antennæ respectively; and d is the distance between them, and λ is the wave-length, whilst K is a constant depending on the unit of length. Also α is a constant depending on the atmospheric absorption.

If heights, lengths, and distances are all measured in kilometres or fractions then $K = 4.251$ and $\alpha = 0.0015$.

Kimura (*The Electrician*, October 18, 1912, p. 51) modifies this formula into

$$I_r = I_s(\text{amps}) \frac{150}{10^6} \frac{h_1 h_2(\text{feet})}{\lambda(\text{metres}) \cdot d(\text{miles})} \epsilon^{-\frac{0.088 d}{\sqrt{\lambda}}}, \quad (2)$$

where h_1, h_2 are in feet, λ in metres, d in miles, and I_S in amperes. The constant ϵ is the base of the Napierian logarithms. Hence if we call this attenuation factor $\epsilon^{-\frac{\alpha d}{\sqrt{\lambda}}} = \frac{1}{F}$, we have $\log_\epsilon F = \frac{\alpha d}{\sqrt{\lambda}}$ or $2.3026 \log_{10} F = \frac{\alpha d}{\sqrt{\lambda}}$.

Example.—A receiving antenna has a height of 200 feet, and 1000 miles away there is a transmitting station with an antenna 400 feet high sending out waves of 6400 metres in length. Assume a receiving current of 40 microamperes is required for good reception. Calculate the sending antenna current required.

Answer.—We have $h_1 = 400$ feet, $h_2 = 200$ feet, $d = 1000$ miles, $\lambda = 6400$ metres, and $I_R = \frac{40}{10^6}$ amperes.

Hence $\sqrt{\lambda} = 80$ and $\frac{d}{\sqrt{\lambda}} = \frac{100}{8}$.

Therefore $\epsilon^{-\frac{0.088d}{\sqrt{\lambda}}} = \epsilon^{-1.1} = \frac{1}{3}$ nearly.

Hence we have by equation (2)

$$\frac{40}{10^6} = I_S \frac{150}{10^6} \frac{400 \times 200}{6400 \times 1000} \times \frac{1}{3},$$

or $$I_S = 64 \text{ amperes.}$$

The formula (1) is only an empirical one based on measurements made over sea up to about 1200 miles and is not to be taken as expressing a physical law of attenuation.

For the discussion of formulæ expressing the value of the received current at various angular distances round a spherical earth, see Prof. A. E. H. Love, *Phil. Trans. Roy. Soc. Lond.* vol. 215 A, p. 105, 1915.

It has been shown by C. Tissot that a receiving antenna on which long electric waves are falling, absorbs a greater fraction of the total energy sent out from the sending antenna than is represented by the ratio of its apparent surface to the whole hemisphere described round the sending station with radius equal to the distance between the stations. This is because the energy moves in on the receiving antenna perpendicularly to the plane containing the electric and magnetic forces in the wave front. As the electric wave sweeps over the antenna and sets up electric currents in it, these last deform the field round the antenna and make a concavity in the lines of magnetic force as they approach the antenna. Hence the antenna is not like a surface exposed to radiant heat or light absorbing only the energy incident on its geometrical projection on the wave front.

To measure the energy captured by an antenna we have to insert near the base a *bolometer* or very sensitive thermal ammeter of known resistance (ρ) which should be large compared with the rest of the antenna, and then we measure by it the mean-square current (I_R^2) at the base of the antenna.

The power given to the receiving antenna is then ρI_R^2 and if the sender is a spark discharger giving N' sparks per second, then the energy received per spark is

$$W_R = \frac{\rho I_R^2}{N'}. \quad . \quad . \quad . \quad (3)$$

It is then interesting to compare this captured or received energy with the energy radiated from the sending antenna.

In considering the proportion of the radiated energy sent

off from a transmitting antenna, which is picked up by a receiving antenna, we must consider the manner in which the radiated energy is distributed in the space around the sending antenna. We have seen, in the case of a Hertzian oscillator of electric moment ϕ, that if we describe a sphere round it with radius r large compared with the wave-length λ, the energy sent out through a zone of angular width $d\theta$ in a time dt is equal to

$$\frac{\phi^2 m^3 n}{2} \sin^3 \theta d\theta \sin^2 (mr - nt) dt, \quad . \quad (4)$$

where $m = \frac{2\pi}{\lambda}$ and $n = \frac{2\pi}{T}$, and θ is the angle between the direction of the radius r and the axis of the oscillator, and T is the periodic time. Hence the energy sent out per complete period is

$$\frac{\phi^2 m^3 n}{2} \sin^3 \theta d\theta \frac{T}{2}, \quad . \quad . \quad . \quad (5)$$

because the mean value of $\sin^2 (mr - nt)$ per complete period is $\frac{1}{2}$.

Since the area of a zone of a sphere of width $d\theta$ and co-latitude θ is $2\pi r^2 \sin \theta d\theta$, it is obvious that the energy radiated per period through every zone of width $d\theta$ is

$$\frac{4\phi^2 \pi^4}{\lambda^3} \sin^3 \theta d\theta. \quad . \quad . \quad . \quad (6)$$

Accordingly it varies as the cube of the sine of the angular distance of the radius vector from the pole of the circumscribing sphere.

The following Table gives the values of $\sin \theta$, $\sin^2 \theta$ and $\sin^3 \theta$, for various values of θ :—

θ.	$\sin \theta$.	$\sin^2 \theta$.	$\sin^3 \theta$.
0	0 0	0.0	0.0
10	0.173	0.03	0.005
20	0.342	0.117	0.040
30	0.500	0.25	0.125
40	0.643	0.413	0.266
50	0.766	0.587	0.449
60	0.866	0.749	0.649
70	0.939	0.882	0.828
80	0.985	0.970	0.955
90	1.000	1.000	1.000

The surface of the sphere described round the oscillator is $4\pi r^2$, and hence the *mean spherical energy* radiated per period from the oscillator is

$$\frac{1}{4\pi r^2} \cdot \frac{16\phi^2\pi^4}{3\lambda^3} = \frac{4}{3}\frac{\phi^2\pi^3}{r^2\lambda^3} \quad . \quad . \quad (7)$$

This corresponds to the mean spherical candle-power of an arc lamp.

This energy is not, however, radiated equally in all directions.

The radiation per square unit of area for each zone of colatitude θ is the quotient of (6) by $2\pi r^2 \sin\theta d\theta$, or is

$$\frac{2\phi^2\pi^3}{r^2\lambda^3} \sin^2\theta \quad . \quad . \quad . \quad (8)$$

Hence the radiation in a horizontal direction, perpendicular to the axis of the oscillator when $\theta = 90°$ is

$$\frac{2\phi^2\pi^3}{r^2\lambda^3} = \left\{\begin{array}{l}\text{Mean horizontal energy radiated}\\ \text{per period through 1 square}\\ \text{unit of area.}\end{array}\right. \quad . \quad (9)$$

Comparing then the

$$\left.\begin{array}{l}\text{Mean spherical radiation}\\ \text{per period per square}\\ \text{unit of area}\end{array}\right\} = \frac{4}{3}\frac{\phi^2\pi^3}{r^2\lambda^3} \quad . \quad (10)$$

with the

$$\left.\begin{array}{l}\text{Mean horizontal radiation}\\ \text{per period per square}\\ \text{unit of area}\end{array}\right\} = \frac{2\phi^2\pi^3}{r^2\lambda^3} \quad . \quad (11)$$

we see that the mean horizontal radiation energy is 1.5 times the mean spherical radiation.

In a particular instance C. Tissot found that the energy radiated per spark from a simple plain sending antenna of Marconi type was 1.3×10^7 ergs per spark.

At a distance of 1000 metres the mean spherical energy per spark per square metre was

$$1.04 = \frac{1.3 \times 10^7}{4\pi(1000)^2} \text{ ergs per spark per square metre.}$$

Hence the mean horizontal energy per spark per metre square at that distance was 1.56 ergs per spark per metre square.

At this distance a receiving antenna was set up 50 metres high and 0.4 cm. in diameter, exposing, therefore, a surface of $5000 \times 0.4 = 2000$ square cms. or 0.2 square metre.

It ought therefore to have captured 0.2×1.56 ergs per spark $= 0.31$ erg per spark.

As a matter of fact the receiving antenna was found to receive 57 ergs per spark, as measured by a bolometer inserted near the base. This shows that it absorbed energy from an area equal to 177 times its own apparent area, or since it was 0.4 cm. in width from an area equal to 35.4 cms. on each side of it. Hence so far from merely absorbing radiation equal to that falling on an area of its own width, it absorbed from a space of more than 1 foot on either side of it.

2. Tuners

The wave energy captured by the receiving antenna is partly dissipated as heat in the antenna itself and in the earth connections, and partly radiated and partly transferred to a closed condenser circuit coupled with the receiving antenna which is called the energy-storing circuit.

This last circuit comprises an adjustable condenser or condensers and certain variable inductances by the adjustment of which the condenser circuit is tuned to the antenna. The receiving antenna has a certain *efficiency of reception* just as the sending antenna has a certain *radiation efficiency*. This last expresses the proportion of the whole energy given to the antenna which is radiated as electric waves, and in the same manner we can denote by the term efficiency of reception the proportion of the whole wave energy which is incident on the antenna which is transferred to the associated condenser circuit as energy of electric oscillations.

The simplest form of energy-storing receiver circuit comprises a single condenser of adjustable capacity which if space permits should be an air-condenser. Condensers having ebonite as the dielectric are very generally used for the sake of compactness but absorb some energy, and as so little energy is captured by the antenna at best we cannot afford to lose any of it. In series with this condenser is an adjustable inductance and one coil of an oscillation transformer or *jigger*, the other coil of which is inserted in the antenna circuit. The mutual inductance of these two last-named coils should be variable, so as to alter the coupling of the circuits.

In the Marconi Company's Tuner there is an intermediate circuit, the object of which is to hinder the transference to the detector circuits of the irregular oscillations set up by atmospheric discharges or so-called "stray" waves.

The scheme of connection of this Marconi Tuner is shown in Fig. 32 and the external appearance in Fig. 33.

There are three adjustable condensers C_1, C_2, C_3, called respectively the antenna, intermediate, and receiving condenser.

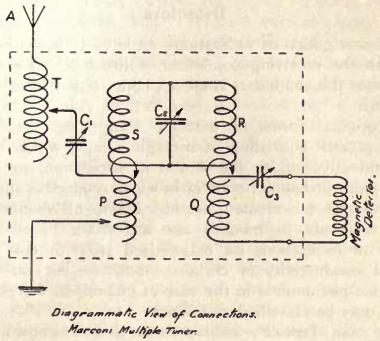


FIG. 32.—Connections of Marconi Company's Tuner.

There are also two pair of oscillation transformers P_1S_1, P_2S_2 of which the coupling can be varied by rotating one of the coils so as to make its axis more or less inclined to that of the other.

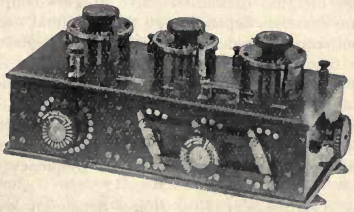


FIG. 33.—External Appearance of Marconi Tuner.

There are also variable inductances by means of which the three circuits can be tuned to the antenna and to the frequency of the incident waves. The oscillations set up in the last circuit, viz. that including the condenser C_3, are those which

operate the detector. For the full theory of the action of two coupled oscillatory circuits the reader must be referred to the author's book, *Principles of Electric Wave Telegraphy and Telephony*, chapter iii.

3. Detectors

For the detection of oscillations we have to insert either in series with the receiving condenser of the tuner or else as a shunt across the condenser some appliance sensitive to electric oscillations.

The original forms of detector comprising the various forms of *coherer* consisting of metallic filings, which became more conductive under the action of oscillation, or various forms of point or contact coherer in which reduced or increased contact resistance is created, are now almost all abandoned.

The detectors in modern use are either based on the influence of oscillations on magnetised iron, or else on the unilateral conductivity of certain conductors, or on certain interference phenomena in the case of undamped waves.

They may be classified as follows :—

In the case of *spark transmitters* the detectors now used are

(1) Marconi's magnetic detector or some modification.

(2) Rectifying crystals or contacts depending on the unilateral conductivity of some crystals, *e.g.* carborundum, or of some contacts between two substances, *e.g.* zincite and chalcopyrite or molybdenite and copper or galena and plumbago.

(3) The Fleming oscillation valve or glow-lamp detector or some modification depending on the unilateral conductivity of a gas between electrodes one of which is incandescent.

In the case of undamped waves produced by *arc transmitters* or *high frequency alternators* the detectors used are

(4) The Poulsen *Tikker*.

(5) The Fessenden *Heterodyne Receiver* depending on the interference between the received persistent oscillations and similar oscillations of slightly different frequency produced locally.

(6) The Goldschmidt *Tone Wheel* depending on the use of a non-synchronous rotating wheel.

For experimental or quantitative work some form of metrical detector must be used, such as a thermoelectric or bolometer detector which can be calibrated with direct currents and then used with alternating currents.

One such form of thermal detector is Duddell *Thermo-*

galvanometer. In this the current to be measured passes through a heater or very thin platinum wire or else a strip of gold leaf. Very close over this heater is suspended by a quartz fibre a thermoelectric couple of bismuth and antimony in the form of a closed loop or rectangle. This is suspended in a strong magnetic field. When a current passes through the heater wire the radiated and convected heat creates a current in the thermocouple circuit, and this is deflected in the magnetic field. A mirror attached to the couple renders small movements visible by reflecting from it a ray of light on to a screen. Another form of thermoelectric detector is that due to the author in which one junction of a thermocouple of tellurium and bismuth is attached to a fine constantan wire through which the oscillation to be detected passes. The heater wire and thermocouple are enclosed in a double-walled vessel of glass exhausted to a high vacuum. The thermocouple is connected by its terminals to a low resistance movable coil galvanometer, and the oscillations to be measured are passed through the fine wire. Another form of thermoelectric detector is the bolometer detector.

In this a couple of rectangles of very fine platinum wire are joined into the arms of a Wheatstone's Bridge and balanced with another pair of resistances by a galvanometer in the bridge circuit. If electric oscillations are sent through one of these rectangles then it is thereby heated and the bridge balance upset. Such thermoelectric detectors are useful for calibrating other metrical forms of detector, but they are not sufficiently sensitive for use in practical radiotelegraphy.

Returning then to practical detectors we shall mention a few matters concerning them.

(1) *The Marconi Magnetic Detector*.—This is one of the most useful, compact, and easily worked detectors and may be combined in one instrument with a tuner. It consists of an endless band or rope of fine silk-covered iron wires which is stretched over a pair of wooden pulleys and is driven round by clockwork at a speed of a few inches per second. In part of its journey it passes through a glass tube wound over with one layer of silk-covered wire through which pass the oscillation to be detected. Over this coil is another coil which is in series with a Bell telephone receiver.

Two permanent horseshoe magnets are placed with similar poles contiguous and close to the moving wire and just against the inner coil embracing it. When oscillations pass through the coil an observer hears sounds in the telephone corresponding

in pitch to the spark frequency of the oscillation creating these oscillations.

The theory of the instrument has been much discussed. It was at first explained by the assumption that the oscillatory magnetic field annulled the hysteresis of the iron. In virtue of this hysteresis that magnetised portion of the moving band is not in the same position with regard to the magnetic poles as if the band is at rest. Hence the oscillations cause a change in the magnetic flux which is linked with the coil in series with the telephone and hence cause an E.M.F. to be created in it.

J. Russell explains it by the increase in the magnetisation of iron resulting from the superposition of an oscillatory field upon a slowly changing or stationary field near a cyclic extreme. The movement of the band is, however, an essential condition of the action, and the energy which creates the sound in the telephone is drawn from the source of energy (clockwork) which drives the band. In any case every train of oscillations which passes through the embracing coil of wire causes a sound in the telephone, and if these trains come regularly with a frequency of 100-500 then a corresponding musical note is heard in the telephone. The continuous sound is cut up into long or short sounds in accordance with the Morse Code by the key in the transmitter circuits.

The Marconi magnetic detector being a series or current operated detector, the oscillatory coil is inserted in series with the receiving condenser in the tuner. The Marconi Company combine together a portable tuner and magnetic detector which is very useful for military and experimental work. This magnetic detector being self-contained and easy to adjust and very portable is much used for military work, although it is not quite so sensitive as some other forms of detector.

For additional information as to the theory the reader is referred to the author's book, *Principles of Electric Wave Telegraphy and Telephony*, chapter vi.

(2) *Rectifying Crystals and Contacts.*—About a couple of dozen or more substances are known, which have the curious property that the contact surface between them when they are taken in pairs or couples under greater or less pressure permits a larger electric current to flow in one direction than in the other under a certain impressed voltage. Hence for this electromotive force, generally a small one, not exceeding a few volts, or fraction of a volt, the contact surface between the two bodies presents a unilateral conductivity.

It is sometimes asserted that there is a unilateral con-

ductivity in certain substances in themselves; but inasmuch as there must be conducting electrodes of some kind it is difficult, if not impossible, to distinguish between so-called "body rectification" and "surface rectification." These dissimilar substances when in contact permit a greater current of electricity to flow across the contact in one direction than in the other under equal but reversed electromotive force.

Hence when the E.M.F. is alternating there is a resultant unidirectional flow which is called the rectified current, and the contact is said to rectify more or less.

If then an electric oscillation is sent across the contact it will be converted into a direct current. Hence if an ordinary galvanometer is placed in series with the contact it will indicate a current, and if a telephone receiver is used, and if intermittent trains of oscillations are used of train frequency above 50 or so, the effect will be the same as if the telephone were traversed by repeated gushes of electricity in the same direction, and it will therefore emit a sound having a pitch corresponding to the group or train frequency of the oscillations.

Hence if the groups are intermitted or cut up by a key in the transmitter into Morse Code signals the telephone will emit corresponding sounds.

The best insight into the operation of these rectifying contacts is obtained by delineating a *characteristic curve* for them. If we apply a small, gradually increasing voltage in one direction and measure for each voltage the corresponding current, the curve plotted with voltage as abscissa and current as ordinate is called the characteristic. If the direction of the voltage is reversed, the branch of the characteristic will then be generally different in form from the first branch. Corresponding to equal positive and negative applied voltages the currents will be different, either greater or less. Hence when an alternating voltage is applied there may be a resulting direct current in one direction or the other. The characteristic curve of certain couples or contacts is remarkable for sudden changes of curvature at certain points.

Some of these rectifying substances are crystals, some nonmetals, some elements, some oxides, sulphides, silicides, or carbides. It is possible to arrange a number of them in a series such that when one of the list is made the positive electrode, the current flowing across the junction to the second material will be great if the second material stands higher on the list than the first, and less if it stands lower.

The following list comprises the names of most of these rectifying substances or couples.

Carborundum.—A carbide of silicon (*SiC*) made in electric furnaces by heating to a very high temperature (1950° C.) coke, sand, and sawdust. The reaction which takes place is expressed by the chemical equation

$$SiO_2 + 3C = 2CO + SiC.$$

The crystallised carborundum presents itself in bluish irridescent or greyish crystals, having a specific gravity of 3.12, and a hardness approaching that of diamond. When held between metallic clips or even if the ends of a crystal are platinised, it possesses marked unilateral electric conductivity,

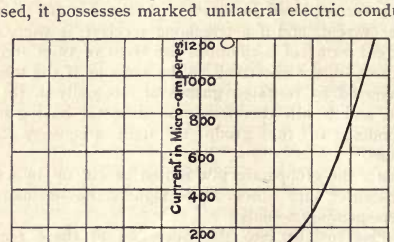


FIG. 34.—Characteristic Curve of Carborundum Crystal.

as first discovered by General H. H. C. Dunwoody (see U.S.A. Patent 837,616, 1906, also British Specification No. 5332 of 1907).

A typical characteristic curve of a carborundum crystal is that shown in Fig. 34.

In the case of carborundum comparatively high voltages can be applied to it, such as 10, 20, or 30 volts, and the rectification improves rather than diminishes. G. W. Pierce found that in the case of one crystal the current was 4000 times greater in one direction than the other under an E.M.F. of 30 volts.

The rectification is reduced by making a very good contact between the crystal and the electrodes, as for instance by platinising both the extremities and using these as electrodes.

It seems clear that with perfectly good platinised contacts

on both surfaces of a slice of carborundum crystal the rectification becomes very small, and this appears to indicate that the ultimate cause of the effect is in some asymmetry of contact surface.

A number of substances have been found, which in crystalline form appear to have the same property, such, for instance, as *Iron pyrites* (FeS_2), *Pyrolusite* (MnO_2), *Molybdenite* (MoS_2); and these all possess asymmetric conductivity.

Hessite, a *telluride* of silver and *anatase* and *brookite*, which are oxides of titanium (TiO_2), have also been found by G. W. Pierce to have rectifying properties similar to carborundum.

F. Braun, in 1874, discovered the similar unilateral conductivity of *psilomelan*, a native hydrated oxide of manganese, and he also found asymmetric conductivity in *copper pyrites*, *galena*, *copper antimony sulphide*, and various other metallic sulphides.

Since that time it has been found that *silicon*, *tellurium*, *graphitic carbon*, *zincite* (*ZnO*) (native oxide of zinc) and *chalcopyrite* and *bornite*, which are native sulphides of iron and copper ($CuFeS_2$), all possess more or less rectifying power when placed between metallic electrodes or, better still, when taken pair and pair in contact. One of the best of these is the combination *zincite-chalcopyrite*, now called the *perikon* rectifier, and the zincite-tellurium combination is also very effective.

The rectifying contacts, properly speaking, are those formed of pairs of substances for which the rectifying property depends to a large extent on the area of the surfaces in contact, on the pressure, and on the voltage applied to the contact.

Thus, for instance, a contact between *silicon* and *steel*, *carbon* and *steel*, *tellurium* and *aluminium*, and between *silicon* and *copper* possesses this rectifying power. Every piece or form of silicon or of carbon, however, does not exhibit it equally. Also if the contact surfaces are large and pressed hardly together, the unilateral conductivity more or less disappears. Thus W. L. Austin found that for a particular silicon-steel contact an E.M.F. of 2.5 volts applied so that the steel was the positive electrode, and the silicon negative passed a current of 25,000 milliamperes across the junction, but if the direction of the E.M.F. was reversed only 6000 milliamperes passed.

In another steel-silicon contact an E.M.F. of 2 volts applied to send a current from *silicon* to *steel* across the junction gave a current of 3.2 milliamperes, but when applied in the opposite

direction so as to send the current from *steel* to *silicon*, the current was 16 milliamperes. Austin also studied the *tellurium aluminium* contact (see "Some Contact Rectifiers for Electric Currents," *Bulletin of Bureau of Standards*, No. 94, vol. v., 1908, Washington, U.S.A.).

For small voltages the rectified current is proportional to the square of the voltage.

The full explanation of this curious rectifying property has not yet been given. An explanation has been based upon the thermoelectric properties of those substances which, in most cases, are very marked, and in the unsymmetrical internal electromotive forces which are created due to the Thomson effect, and the Peltier effect in these substances (see W. D. Eccles, *Proc. Phys. Soc. Lond.* vol. xxv. p. 273, 1913).

On the other hand it may be remarked that most of the good rectifying substances are oxides, or sulphides, one at least of the two being generally a sulphide in the most effective combinations. Thus we have the zincite (oxide)-chalcopyrite (sulphide) as one of the best, and the molybdenite (sulphide)-copper as another. Likewise the galena (sulphide)-plumbago as a third. Also other good rectifiers involve a metal and a non-metal in contact. For example, tellurium-aluminium, silicon-steel, carbon-steel, one being a much better conductor than the other.

The author of this book has also discovered that in these rectifying couples one of the two bodies is much more photo-electric than the other, and loses electrons more easily under the action of ultra-violet light. The following are a few notes on the rectifying power of various combinations from experiments in the author's laboratory.

1. *Zincite-Chalcopyrite Contact = Perikon Detector.*—With very light- or medium-pressure contacts the greater current flows when the *zincite* is connected to the *negative* pole of the battery. When the pressure is increased considerably, giving larger areas of contact between the crystals, the same direction holds good for all voltages down to $\frac{1}{500}$ about of a volt. For lower voltages the direction of the maximum current may be reversed.

2. *Molybdenite-Copper Contact.*—With light pressures and small contact areas the greater current flows for all voltages with the *molybdenite negative.* With heavy pressures and larger contact areas the maximum current may be in the opposite direction. Using flat copper contacts pressing on the molybdenite, the above reversal, that is the maximum current

flowing with the *copper negative*, may take place for voltages as high as two volts, but if the pressure is removed so that the copper only touches on a few points the maximum current flows as above stated when the molybdenite is negative.

3. *Zincite-Bornite Contact.* — With light pressures the direction of the maximum current is with the *zincite negative* for all voltages, except very small ones. With heavy pressures giving larger areas of contacts, the current in the reversed direction with the bornite negative is greater, for all voltages below about 0.1 volt.

4. *Zincite-Copper Contact.*—The general direction of the maximum current for light pressures is with the *zincite negative*. With heavy pressures and larger contact areas, the larger current flows in the reversed direction, that is with the *copper negative*, for very small voltages of about 0.10 volt.

5. *Zincite-Galena Contact.*—The maximum current is with *zincite negative*, but for very small voltages the greater current is in the reverse direction, but the unilateral conductivity in either direction is not well marked.

6. *Molybdenite-Chalcopyrite Contact.* — Shows good unilateral conductivity, the maximum current flowing when the *molybdenite* is *negative*.

7. *Molybdenite-Bornite Contact.* — Gives good unilateral conductivity; the maximum current under all conditions is with *molybdenite negative*.

8. *Iron Pyrites-Zincite Contact.* — Gives very good unilateral conductivity for all pressures, and best for fairly heavy pressures. The direction of maximum conductivity is with *zincite negative*.

9. *Iron Pyrites-Molybdenite Contact.*—Shows very good unilateral conductivity, for voltages of about 0.5 volt and with fairly high pressures. The direction of maximum current is with the *molybdenite negative*.

10. *Silicon-Zincite Contact.*—Shows very good unilateral conductivity for voltages above about $\frac{1}{2}$ a volt, but below that practically none. The direction of maximum conductivity is with the *zincite negative*.

11. *Tellurium-Zincite Contact.*—Gives very good unilateral conductivity in all cases. The maximum current is with *zincite negative*. The resistance of this contact is very high.

12. *Tellurium-Molybdenite Contact.* — Gives a good unilateral conductivity, especially with fairly light contacts. In all cases the direction of maximum current is with the *molybdenite negative*.

13. *Tellurium-Silicon Contact.*—Shows a fairly good unilateral conductivity, but only with light contact pressures. The direction of the maximum current is always with the *silicon negative.*

In connection with these rectifying contacts the question arises; assuming we know in which direction a given current passes under any given electromotive force for two substances *A* and *B*; and for two substances *B* and *C*, whether we can predict the result when the substances *A* and *C* are in contact. This question has been partly answered by Mr. A. F. Hallimond for ten selected rectifying substances, giving therefore $45 = \frac{1}{2}(9 \times 10)$ pairs of possible contacts.

He found he could arrange them in such an order that the largest current flows under a given electromotive force from the one lowest in the list to the one highest in the list across the contact. Thus Mr. Hallimond's order for the following 10 substances is—zincite, brookite, molybdenite, zinc, galena, iserite, copper, chalcocite, chalcopyrite, tellurium.

Thus if we take zincite-chalcopyrite and connect zincite to the negative and chalcopyrite to the positive pole of a cell, the current across the junction will be larger than if the battery is reversed. The same will hold good for the molybdenite-copper contact. Hence zincite is negative to nearly all other rectifiers and the zincite-tellurium is a very effective rectifying contact.

(3) *The Fleming Oscillation Valve.*—This detector consists of a carbon or metallic filament glow lamp having a plate of metal sealed into the bulb, which is carried on a third or insulated terminal. The plate generally takes the form of a cylinder of nickel or copper which surrounds but does not touch the filament. When the filament is rendered incandescent by a current from a battery of a few cells the space between the filament and the cylinder filled with rarefied air becomes conductive in virtue of negative electrons present in it which are given off from the filament. It therefore conducts electricity but only in one direction, viz. negative electricity can pass from the filament to the plate. The appliance acts therefore as an electric valve. Hence its name. It is used in series with a galvanometer or a telephone to rectify electric oscillations. The usual connection is as follows :—

The receiving antenna has a condenser circuit coupled to it or else the Marconi form of tuner already described. The Fleming oscillation valve is connected across the terminals of the receiving condenser so that the plate of the valve is connected to one condenser terminal through one coil of

an oscillation transformer, the other circuit of which is in connection with a telephone. The negative terminal of the

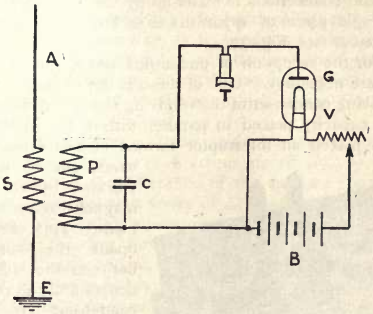


FIG. 35.—Scheme of Connections for Reception by Fleming Oscillation Valve.

filament of the valve is connected to the second terminal of the receiving condenser. There is generally a rheostat for

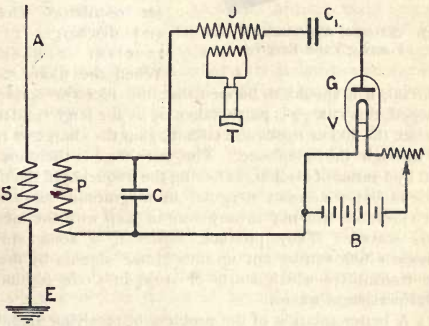


FIG. 36.—Marconi's Scheme of Circuits when using Fleming Valve.

regulating the current from the storage battery which ignites the filament. The connections will be understood from Figs. 35 and 36.

The glow lamp is generally a 10 or 12 volt lamp, so that 6 small storage cells can incandesce it. The whole arrangement of valve and connections is made up by the Marconi Company into a single piece of apparatus in a box which is called a valve receiver (see Fig. 37).

(4) For the reception of undamped waves special types of receiver are necessary. One of these is the Poulsen "tikker." The receiving condenser of the receiving circuit has a condenser of larger capacity placed in parallel with it, but in the shunt circuit is placed an interrupter formed of two crossed gold wires, one of which is vibrated by an electro-magnetic buzzer arrangement. This closes and opens the connection between the large condenser and the receiving condenser. In parallel with the large condenser is a telephone.

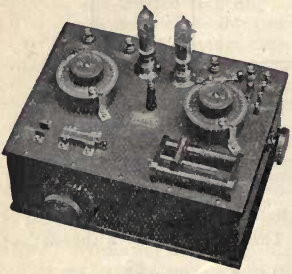


FIG. 37.—External Appearance of Marconi-Fleming Valve Receiver.

The action is as follows: If continuous waves fall on the antenna they set oscillatory charges and discharges in the receiver condenser. When the tikker closes the circuit, if it should so happen that the receiver condenser is charged, this charge is partly taken up by the large condenser, and when the tikker opens the circuit again the charge so taken flows through the telephone. The telephone is therefore traversed by a series of discharges having the frequency of the tikker vibrations but somewhat irregular in magnitude because the tikker vibrations are not in any way in step with the incident electric waves. They produce, however, a sound in the telephone which can be cut up into Morse signals by the key in the transmitter which starts or stops in Code fashion the train of persistent waves.

(5) A better solution of the problem of receiving signals by undamped electric waves is by the Fessenden heterodyne receiver.

If we have undamped waves falling on a receiving antenna we cannot employ a rectifying contact to receive signals with the same connections as for undamped waves, because the

rectifier would merely cause a continuous current to flow through the telephone, and there would be no audible sound and only short "ticks" for each Morse signal. If, however, we have a local high frequency alternator, the speed of which we can control so as to produce a current having a frequency a little greater or a little less than that of the received current, we can cause this to induce in the antenna a superimposed set of oscillations which create *beats* with the message bearing oscillations.

Thus if the frequency of the incident waves is 30,000, and the local alternator gives a frequency of 29,500, the *beat frequency* will be the difference of the two, viz. 500. These beats are equivalent to a series of *damped* oscillations in close sequence. Accordingly, we can couple inductively to the antenna an ordinary tuner and damped wave receiver and detector such as a crystal, rectifying contact, or Fleming valve detector, and we then hear in the telephone sounds having the pitch of the beat frequency which can be cut up into Morse Code signals by the manipulation of the key in the transmitter at the sending end.

(6) Another device for receiving signals by undamped waves is the Goldschmidt tone wheel. In this appliance a metal wheel has teeth cut in its periphery which are filled in with insulating material. A contact brush presses against the edge. The wheel is driven at a uniform but controlled speed by an electric motor. If this wheel and brush are inserted in series with a telephone placed as a shunt across a condenser in a circuit coupled to and tuned with an antenna receiving undamped waves, then if we suppose the wheel standing still with the brush in contact with a metal tooth of the wheel, the telephone would be traversed by undamped oscillations, provided waves of similar type were falling on the antenna. These would create no sound in the telephone if their frequency was above 20,000. If then we suppose the wheel set in rotation with such a speed that the brush makes contact with a tooth when the electric impulses are in one direction, and with an insulating section when they are in the opposite direction, then the telephone would be traversed by unidirectional impulses and would give out no sound.

If, however, the wheel is not rotating at synchronous speed but a little faster or slower there will be a continual fluctuation of current through the telephone which may, by adjusting the speed of the wheel, be made to have any required frequency. These *beats* will produce a sound of the

beat frequency, and just as in the case of the heterodyne receiver the manipulation of the key at the transmitting end will cut up the sound into Morse Code signals. The receiving operator can control the speed of wheel so that the difference between the number of interruptions of the brush contact per second and the frequency of the undamped oscillations is some convenient number near to 400 or 500.

4. Comparison of Detectors with Regard to Sensitiveness

In testing or comparing the sensitiveness of detectors of electric oscillations it is essential to have some means of producing feeble oscillations of known frequency and damping. This is best done by a buzzer oscillation circuit.

A buzzer is an electromagnetic interrupter which opens and

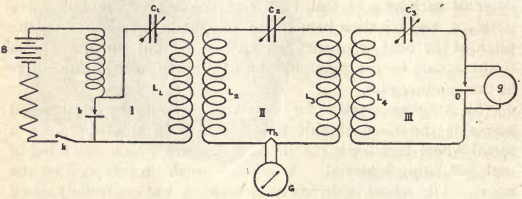


FIG. 38.

closes a circuit very rapidly with known frequency. It is operated by a single storage cell with resistance in series, and a high resistance carbon filament lamp can be connected across the contacts to quench the spark. The contacts are also short circuited by an air condenser of variable capacity in series with an inductance and resistance. When the buzzer contacts open the cell charges the condenser, and when they close again the condenser discharges with oscillations through the buzzer. By varying the capacity and inductance we can control the frequency, and by varying the resistance we can regulate the damping.

This buzzer is employed to set up oscillations in another circuit coupled to it, which second circuit contains a condenser and inductance, and can be adjusted to imitate in oscillation constant any required antenna. This second oscillation circuit contains a thermoelectric ammeter in circuit, such, for instance, as a Duddell thermoammeter. Coupled to this second circuit

is a third oscillation circuit (see Fig. 38), which also contains a capacity and inductance, and by variation of these it can be tuned to the other two coupled circuits. In this tertiary circuit is placed the detector (rectifier or other form) which it is desired to test.

If we call the buzzer circuit No. 1, and the second or thermoammeter circuit No. 2, and the third or detector circuit No. 3, we can proceed as follows :—

Arrange the buzzer circuit to give oscillations of required frequency N and decrement D. If C_1 is the capacity, L_1 the inductance, and R_1 the resistance of this circuit, then we have

$$N = \frac{5 \times 10^6}{\sqrt{C_1 L_1}}, \quad . \quad . \quad . \quad (12)$$

provided C_1 is expressed in microfarads, and L_1 in centimetres. Also we have

$$D_1 = \frac{R_1 \times 10^9}{4NL_1}, \quad . \quad . \quad . \quad (13)$$

where D_1 is the decrement per half period, and R_1 is the resistance of the circuit in ohms, and L_1 is in centimetres. Then tune the secondary and tertiary circuits by altering their capacities and inductances until the currents in them are a maximum. Arrange a sensitive galvanometer as a shunt across a crystal or contact detector, say a *Perikon* placed in the tertiary circuit which is untuned.

Then if we vary the coupling between circuits 1 and 2 we shall obtain varying thermocurrents in 2 and galvanometer currents in 3. It is found that the deflections of the galvanometer shunted across the detector in circuit 3 vary as the square of the currents measured in circuit 2.

If, then, we plot a curve which shows the variation of the square of the current in 2 in terms of the coupling between 1 and 2, we can compare various detectors inserted in circuit 3 as follows: Substitute for the galvanometer a pair of head telephones. Alter the coupling of circuits 1 and 2 until with any given detector the telephone gives a just audible sound. We can then find from the previously obtained curve or curve produced, the square of the current in circuit 2 corresponding to that coupling of 1 and 2 and least audible sound in 3. Trying this for various detectors we can say that the sensitiveness of the detector is inversely proportional to the square of the current in circuit 2 which corresponds to such a coupling of 1 and 2 as produces the just audible sound

in the telephone in circuit 3. Another but less satisfactory method of comparison is by shunting the telephone until the sound heard just becomes inaudible. The sensitiveness of any detector is then obtained as follows: After adjusting the circuits so as to give sounds in the telephone shunted across the detector, or it may be in series with the detector, a shunt resistance is applied to the telephone, and varied until the signals just become inaudible. Then if R is the resistance of the telephone, and r that of the shunt, we can take the value of $\frac{r+R}{r}$ as representing the sensitiveness of the detector, because the more sensitive the detector the more completely will it rectify the oscillations, and the smaller will have to be the resistance of the telephone shunt required to just annul all sound. This annulling shunt resistance is not, however, sharply marked, and hence the shunted telephone method is not a very accurate one. There is also considerable doubt whether the shunted telephone method gives values which are metrical; that is strictly proportional to the current or to the square of the current in the receiving antenna. Hence it must be used with caution.

CHAPTER XI

PRACTICAL INFORMATION FOR WIRELESS OPERATORS

THE practical part of wireless telegraphy cannot be learnt from books but only in the practice school and wireless station. Nevertheless it may be useful for the operator, out of reach of other books, to have certain practical information collected for reference. Accordingly in this Chapter we give some notes on

1. Codes and signals.
2. The care of storage cells.
3. The care of wireless apparatus.
4. A glossary of terms used in wireless telegraphy.

1. Codes and Signals

By far the larger part of wireless telegraphy is conducted by means of the International Morse Code, in which each letter or numeral is signified by a group of dot, or short, and dash, or long, signals. These may be either printed in some way on tape or else heard as long and short sounds in a telephone.

Although the International Code is universally adopted in Europe there is a rival to it in the American Morse Code, the use of which is general in the United States of America, and as this latter was introduced before the general convention which unified European codes, it still survives.

The American Code is slightly more rapid than the International in working, but as the American Code makes use of certain spacing within letters such as **c**, **o**, **y**, and **z**, it requires more attention to exact spacing within the letters than does the International Code, as dots are not always at dot space intervals.

The American **l** is a very long dash equal to six dots and

the American **zero** is a still longer nine dot dash.

These codes are as follows :—

International Morse Code

Rules for lengths and spacing: A dash is equal in length to 3 dots. The space between dots and dashes is equal to 1 dot. The space between letters is equal to 1 dash. The space between words is equal to 5 dots.

The International Morse Code Alphabet

a ·—	h ····	q ——·—
ä ·—·—	i ··	r ·—·
à ä ·——·—	j ·———	s ···
b —···	k —·—	t —
c —·—·	l ·—··	u ··—
ch ————	m ——	ü ··——
d —··	n —·	v ···—
e ·	ñ ——·——	w ·——
é ··—··	o ———	x —··—
f ··—·	ö ———·	y —·——
g ——·	p ·——·	z ——··

The International Morse Code Numerals

1 ·————	6 —····
2 ··———	7 ——···
3 ···——	8 ———··
4 ····—	9 ————·
5 ·····	0 —————

Bar for fractions thus /.

—··—·

Punctuation

Full Stop	(.) ·—·—·—
Semi-colon	(;) —·—·—·—
Comma	(,) ——··——
Colon	(:) ———···
Note of Interrogation or Query	(?) ··——··
Note of Exclamation	(!) —·—·———
Apostrophe	(') ·————·—
Hyphen	(-) —····—
Brackets or Parenthesis	( ) —·——·—

Quotation or Inverted Commas ("")	·—··—·
To call up Signal before Message	—·—·—
Double Dash separating Addresses or Signatures from Message	—···—
Understood	···—·
An Error or Erase	··········
End of Transmission	·—·—·
Invitation to transmit or ready to receive	—·—
Wait	·—···
End of Work, Sign off	···—·—

In first memorizing these Morse Code signals, the beginner will find it an advantage to divide the alphabet into groups of signals each wholly of dots or of dashes. Thus we have in the International Code :—

e = · **i** = ·· **s** = ··· **h** = ····

The word **eish** therefore memorizes this set of dot signals and the student can remember it by the word **eish** = 1, 2, 3, 4 dots.

Again we have a dash group as follows :—

t = — **m** = —— **o** = ——— **ch** = ————

and **tmoch** = 1, 2, 3, 4 dashes memorizes this group.

Then another group is formed by putting a *dash* after 1, 2, 3 dots, or a *dot* after 1, 2, 3 dashes, thus :—

a = ·— **u** = ··— **v** = ···—
n = —· **g** = ——· **ö** = ———·

Certain letters are the reverse of each other, thus :—

a = ·— and **n** = —·
also **b** = —··· and **v** = ···—
and **d** = —·· and **u** = ··—
and **f** = ··—· and **l** = ·—··

By a few devices of this kind the bulk of the letters can be soon learnt. Also other signals such as those for **j** and **p** can be remembered by associating with them the words **jam** and **pan** and then remembering that the signal for **j** = ·——— is that of **a** followed by **m**, and that of **p** = ·——·— is that of **a** followed by **n**. Likewise **f** and **l** are recalled by **fin** and **led** since **f** = **i** + **n** = ··—· and **l** = **e** + **d** = ·—··.

The signals for **x** and **y** can be memorized by the words **tit** and **tat**, since **x = t + i + t** and **y = t + a + t**.

The following is the American Morse Code. Note the spacing in letters **c, l, o, r, y, z,** and **&**.

The American Morse Code

The Américan Morse Code differs in some signal letters from the International Code. It is therefore given here for comparison.

a · —	o · ·	1 · — — ·
b — · · ·	p · · · · ·	2 · · — · ·
c · · ·	q · · — ·	3 · · · — ·
d — · ·	r · · ·	4 · · · · —
e ·	s · · ·	5 — — —
f · — ·	t —	6 · · · · · ·
g — — ·	u · · —	7 — — · ·
h · · · ·	v · · · —	8 — · · · ·
i · ·	w · — —	9 — · · —
j — · — ·	x · — · ·	0 —
k — · —	y · · · ·	(.) · · — — · ·
l — — —	z · · · ·	(?) — · · — ·
m — —	& · · · ·	(!) — — — ·
n — ·		

2. Care of Storage-Cells

As storage-cells are used in connection with emergency transmitter sets on board ship, and also used in land stations to supply current for various purposes, it is essential that a wireless operator should be acquainted with the rules for charging and for keeping in order such cells. The neglect of these precautions leads to the rapid deterioration of the battery.

A storage-cell, commonly called a secondary cell or accumulator, is generally a lead cell, that is, one made with lead plates, although iron-nickel cells made in accordance with Edison's method are sometimes used.

The lead cell consists of two lead plates or grids which are placed in a glass or lead vessel filled with dilute sulphuric acid. One of these lead plates is covered on the surface with peroxide of lead and the other with reduced or spongy lead. The spongy lead when prepared in a certain manner is nearly as oxidizable or attacked by sulphuric acid as zinc, and therefore

this plate acts like the zinc plate or rod in an ordinary primary battery.

There are two ways in which the lead plates may be prepared.

(i.) We may apply to a lead grid oxide of lead or litharge made into a paste with dilute sulphuric acid, and this paste may be contained in cavities or perforations in the plates. This paste can then by means of a current of electricity be transformed into peroxide of lead on one plate and spongy lead on the other plate. Such plates are called *packed* or *pasted* plates, and the process of reducing or oxidizing them is called *formation.*

(ii.) The lead plates may be prepared in such form as to have a very large surface, and by using these plates as the electrodes for the decomposition of a certain electrolyte, and frequently reversing the direction of the current with intervals of rest between, the plates may be converted superficially into peroxide of lead and spongy lead respectively. Such a process is called Planté's process, whilst the pasted plate is called a Faure plate.

Whichever form of plate is employed the final result is that we have a plate of lead covered with spongy lead as the negative plate, and a plate of lead covered with peroxide of lead as the positive plate placed in an electrolyte consisting of dilute sulphuric acid of a certain density, the liquid being contained in a glass or lead-lined wooden box. This cell then gives a current when we connect the positive plate by a wire with the negative plate. As the cell discharges, the spongy lead and the lead peroxide are both converted into sulphate of lead, and when the process is complete the cell is said to be discharged.

The chemical change which takes place may be represented by the equation—

$$\begin{matrix} - & & + \\ Pb + n(H_2SO_4) + m(H_2O) + PbO_2 \text{ (fully charged)} \\ - & & + \\ = PbSO_4 + (n-2)(H_2SO_4) + (m+2)H_2O + PbSO_4 \\ \text{(fully discharged).} \end{matrix}$$

This implies that there are before discharge n molecules of strong or neat sulphuric acid (H_2SO_4) and m molecules of water (H_2O) present for each molecule of spongy lead Pb and lead peroxide PbO_2. Then the process of discharge consists of sulphating the plates with consequent dilution of the acid.

If then a current of electricity from a dynamo is passed through the discharged cell entering at the positive plate, the lead sulphate on each plate will be broken up and the spongy lead and lead peroxide reformed. This process is called charging the cell. It results in giving back energy to the cell, which can then be recovered in part as a discharge current. Each cell generally consists of a number of positive plates and negative plates alternated, and all the positives are welded to one lead bar and all the negatives to another, and these two bars form the terminals of the cell. A number of cells can be joined in series by welding or clamping the positive bar of one cell to the negative bar of the next cell, and so on for the series.

When the cells are discharging each cell gives an E.M.F. of about 2 volts. In order to charge the cells we have to provide a direct current dynamo capable of giving an E.M.F. of at least 2.5 volts for each cell in the storage battery to be charged.

Rating of Storage-Cells.—When charged a cell can give out a certain amount of electricity which is reckoned in *ampere-hours* (A.H.), each unit equal to a current of 1 ampere flowing for 1 hour. The useful capacity of a cell is therefore reckoned in ampere-hours. This is called the A.H. capacity.

Also a cell can give a certain current in amperes depending on the external and internal resistance and the E.M.F. of the battery. This is called its ampere-rate of discharge.

The A.H. capacity depends upon the ampere rate of discharge. Thus if a certain cell has an A.H. capacity of 100 A.H. at 10 ampere rate of discharge it will have much less than this capacity at 20 ampere rate of discharge, and somewhat greater capacity at a 5 ampere rate of discharge.

It is usual to rate the cell capacity at so many ampere-hours at a certain number of hours' discharge. Thus if a cell has 100 A.H. capacity at 10 hour rate, this means that if a current of 10 amperes is taken from it steadily for 10 hours, it will give 100 A.H. before the terminal voltage of the cell falls below 1.8 volts.

The normal time of discharge is taken as 9 hours or a working day. In addition to the A.H. capacity we may consider the quantity of energy taken from the cell. This is reckoned in *watt-hours* (W.H.), and we have therefore a certain W.H. capacity as well as A.H. capacity.

In charging the cell we have to put into it a certain quantity of electricity and a certain energy, and we have

therefore to consider the *efficiency* of the cell which means the ratio expressed as a percentage of that which is given out to that which is put into the cell. We have therefore an *ampere-hour efficiency* and a *watt-hour efficiency.* The former may be about 85 to 90 per cent and the latter 70 to 80 per cent. The reason why the latter is less is because the charging voltage of the cell is always greater than the discharge voltage. The ampere-hour and watt-hour efficiency also both depend on the rate of discharge. The normal rate of discharge is about 0.04 ampere per square inch of positive plate surface, and the weight ratio about 4 ampere-hours per *lb.* of plates, reckoning positive and negative together.

Setting up Cells and Charging.—When the cells have to be set up into a battery, unless this is quite a small or portable one, they must be arranged on shelves or on wood stands in a well-ventilated room. The floor should be cement and laid with a cant, and with surface drains so that any acid spilt may run to a drain. The room should be well lit by a window, and provided with electric light. If the cells are in glass boxes each must stand in a wooden tray filled with sawdust. The tray is supported on four glass insulators. The connections between the various cells are best made by welding the lead bus bars together.

A continuous current shunt wound field dynamo or supply circuit has to be provided which is capable of giving 2.5 volts E.M.F. for each cell to be charged. This voltage must be capable of regulation by resistance inserted in the field (see Fig. 39), so that the cells may be charged with a constant current. The voltage, therefore, must be variable, from about 1.8 volts per cell to 2.5 volts per cell.

If supply is taken from a public electric supply at constant voltage this constant current can best be supplied by an *automatic booster*, which is a form of dynamo driven by a motor from the supply mains. It adds to the supply voltage just the proper voltage to keep a constant current flowing into the cells right through the time of charging. In the case of small cells, where the charging current is only 1 or 2 amperes, they may be charged off a ship's lighting circuits or a town supply mains through a resistance which may consist of incandescent lamps arranged in parallel, but must be capable of adjustment as to resistance. Thus a dozen metallic filament glow-lamps may be connected in parallel between two wires. The lamps can conveniently be mounted in switch sockets on one board. This group of lamps must be inserted between

the cells to be charged and the supply mains, and an ammeter also placed in series to show the current. Thus, suppose we wish to charge a small storage battery of 6 cells off a direct current supply circuit, having a voltage of 220, the cells taking 2 amperes to charge, we must mount up on the board 12 to 14 30-watt 200 volt glow-lamps, each taking 0.15 ampere, and join up all these lamps in parallel, and insert the group in series with the cells across the supply mains, taking care that the positive pole of the battery is in connection with the positive terminal of the supply. The battery will then charge with a current of about 2 amperes, and this may be allowed

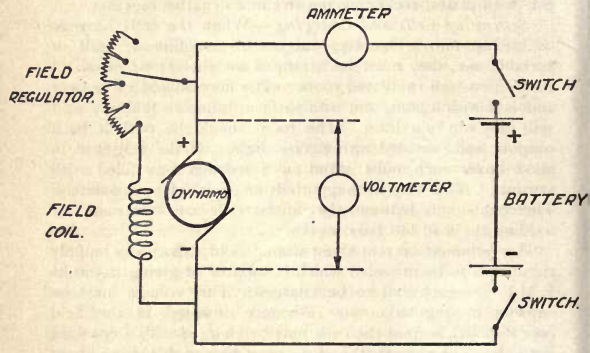


FIG. 39.—Scheme of Connections for a Dynamo charging a Storage Battery.

to run for 10 hours or so until the cells are charged. In the case of larger cells the process of charging must be watched by noticing:

(1) The gradual increase in voltage of each cell as charged. This varies from 2.1 at the beginning to 2.6 volts at the end of the charge. This must be determined by the voltmeter.

(2) The gradual rise of density of the electrolyte.

The electrolyte must be made with pure sulphuric acid manufactured from brimstone, not pyrites, and must be quite free from arsenic, iron, hydrochloric acid, or nitric acid. The specific gravity of the strong acid must be 1.840, and it must be diluted by the addition of the acid to distilled water in the proportion of 1 part of acid to 4 of water, so that when cool

at 60° F. it has a density of 1.215. In this state it is put into the cell, the plates being charged already. As the plates discharge the sp. gr. falls to about 1.17 and rises again as charge.

(3) The plates when fully charged should be free from white spots or patches (lead sulphate). The positive plate should be a uniform chocolate colour, and the negative a clean grey.

(4) At the end of the charge the electrolyte is filled with small gas bubbles and becomes opaque and milky. The plates are fully charged when gas bubbles come off freely from all plates.

The charging requires therefore tests by the cell voltmeter and hydrometer, and inspection by a lamp. In the case of cells in wooden boxes we have to rely chiefly on the voltmeter and hydrometer.

In ships' wireless plants the batteries used are generally 10-ampere cells, having about 70 A.H. capacity at the 10-hour rate.

They are 9 plate cells in lead-lined wooden boxes, and are charged either by a special dynamo in the engine-room or from the ship's lighting circuits.

The following information has been kindly furnished by the Electrical Power Storage Company, Ltd., of Mill Wall, London, E.C., with regard to the working of their 9 plate E.P.S. cells for Ships' Wireless Telegraph Installations.

SHIPS' WIRELESS TELEGRAPH INSTALLATIONS

INSTRUCTIONS FOR E.P.S. E.S. TYPE BATTERIES (9 PLATES)

Initial Charge

1. The acid must not on any account be put into the cells until all is ready for charging.

2. The sulphuric acid must be chemically pure, the quality known as accumulator acid being most suitable, and must be diluted with pure water to a specific gravity of 1.200 when cold and tested by hydrometer.

3. *Mixing Acid.*—If the solution for use consists of "brimstone" acid of 1.84 specific gravity, it must be slowly poured into pure water in a separate and clean lead or glass mixing vessel, well stirred and allowed to cool before the specific gravity is taken. The mixing requires to be done with great care on account of the heat generated. The proportions by measure are about 21 parts of water to 4 parts of acid. The acid must always be added to the water, never the

water to the acid. Distilled water should always be used. The solution should stand for several hours after mixing and should be occasionally stirred before it is put into the cells.

4. *Filling up.*—When everything is ready for charging, the acid is to be put into the cells to a height of not less than half-inch above the tops of the plates.

5. *First Charge.*—Charging must be commenced immediately after the acid is added, and continued if possible until the first charge is completed. On no account should the current be stopped during the first twelve hours, nor should the cells be used to supply current until the acid has turned milky in consequence of the vigorous evolution of the gases from both positive and negative plates, nor until the specific gravity has ceased to rise for at least two hours, when it should not be less than 1.215. Should one set of plates commence to gas before the other the charging rate should be reduced to about half. It may be mentioned that the specific gravity of the acid will fall when it is first put into the cells, and may not commence to rise for a considerable time after the charging has started. The total duration of first charge at normal rate may be as much as forty to fifty hours, and the above directions should be strictly adhered to.

6. *Rate of First Charge.*—The normal rate of first charge should be 6.5 amperes, and the voltage when fully charged (measured while current is on) should be about 2.65 volts per cell.

The normal rates of charge and discharge for these cells are as follows :—

CHARGE.		DISCHARGE.		
Normal current . .	10 amps.	7 amps. for 10 hours		Normal output.
Maximum current .	15 ,,	9 ,, ,, 7 ,,		
		16 ,, ,, 3 ,,		
		36 ,, ,, 1 ,,		Emergency output.

Working Instructions

1. *Charging.*—The cells should always be kept as fully charged as possible. It should be remembered that completely discharging a battery has the effect of shortening its useful life. The normal rate of charge for these cells is 10 amperes. This may be increased on emergency to 15 amperes, but under no circumstances after the plates have commenced to evolve gas must the rate of charge exceed 10 amperes. The charge is to be considered finished when in all the cells gas is freely liberated from both positive and negative plates, the voltage being not less than 2.5 per cell. In no case should the charging be prolonged after the voltage has risen to 2.65 per cell with current at normal rate. In addition to the above charging, once only in every fourteen days the charge should be continued at 10 amperes until the plates have gassed freely for at least 30 minutes, and the voltage has risen to 2.65 to 2.7 per cell.

2. Before cells are connected to the charging source care must be taken to see that the proper connections are being made. The positive

lead of the charging circuit must be connected to the positive + (generally painted red) terminal or lug of the cells, and the negative lead to the negative − (generally painted black) terminal or lug of the cells. If a number of cells are to be charged at once, and the voltage of the charging circuit is sufficient, *i.e.* equal to about 2.2 volts per cell at the commencement and 2.7 volts per cell towards the finish, the cells should be connected in series (*i.e.* negative of one cell to positive of next cell and so on till all cells are joined up). Careful attention must always be paid to the charging, particularly to see that the charging voltage does not diminish, or the current may become reversed and the cells caused to discharge thereby.

3. The emergency rate of discharge should not be exceeded.

4. *Specific Gravity.*—The specific gravity of the acid when the cells are fully charged should be from 1.205 to 1.220. The gravities of the cells should always be taken after a charge in order to ascertain whether the plates are in order. A low specific gravity at the end of a charge would indicate (provided it is not due to excessive topping up with water) that the cell is needing attention. Acid should never be added to cells except under advice from the Company.

The plates must be kept covered with liquid, any loss being made up by the addition of pure water only.

5. *Irregularity.* — Any cell in which both positive and negative plates do not gas freely towards the end of the charge should be carefully examined. Contacts between the plates caused by pieces of paste, scale, etc., or foreign material, should be immediately removed, and the cell cut out of circuit, if on discharge, and put on charge again as soon as possible. It will generally be found that one or two charges will restore the cell to its proper condition.

6. *Disuse.*—If it is required to leave the cells out of work for any length of time, they must previously be fully charged, and care taken to see that the plates are well covered with liquid. If possible a short charge should be given, say once a fortnight, till the cells gas freely, by which treatment the cells will be kept in order for any reasonable length of time.

For larger E.P.S. cells the Electrical Power Storage Company, Ltd., issue the following instructions which by their kind permission are reproduced here.

Instructions for Attendant

Charging and Discharging

1. Keep the battery as fully charged as possible and do not leave it in a partly discharged condition for any length of time.

2. Never discharge after the final permissible voltage corresponding to any current has been reached. Always stop well before this if possible.

3. Charge *immediately* cells fall to the final voltage corresponding to the actual current flowing.

NOTE. — Before closing the charging circuit see that the charging voltage is somewhat higher than the battery voltage.

4. Charge ordinarily at normal rate only, and at no other rates unless the E.P.S. Co. agree.

5. Charge at maximum rate in emergency only and never, unless for special reasons, after cells commence to gas.

6. Never charge at less than the minimum rate specified.

7. Carefully watch the pilot cells and do not consider the charge finished until gas is freely liberated in large bubbles from *both* plates in the pilots. The voltage at this point, with normal charging current flowing, will be from 2.65 to 2.75, and the specific gravity from 1.200 to 1.215. *Guard against excessive overcharging.*

8. On every charge inspect each cell in the battery, and see that all are similar in condition to the pilots as regards commencement and amount of gassing.

9. Examine very carefully any cell gassing to a less extent than the pilots, and if any short circuits are found, remove them immediately by means of a slip of wood or other insulating material.

10. After short circuits have been cleared, bring the slow cells into line by charging them with a milking booster, if of the burnt-up type, or by cutting them out on the following discharge, repeating this process, if necessary, if of the bolted-up type.

Overcharge

11. Once weekly, in the case of batteries getting a daily charge, the charging rate should be reduced to half-normal at the point when the cells are usually considered fully up and the charge continued for half an hour extra. Where cells are charged two or three times weekly this may be done once a fortnight.

Similar treatment should be given if through any cause it has not been possible to charge the battery fully on a preceding occasion, or if the battery has been unavoidably left in a discharged condition, or when, in emergency or by accident the specified capacities have been exceeded, or after a capacity test.

Electrolyte

12. *Never add acid unless advised by the E.P.S. Co.* When the E.P.S. Co. sanction its use it should be added to the tops of the cells towards the end of a charge.

13. *Never use acid or water in the cells unless a sample has been approved by the E.P.S. Co.*

14. Never add acid to a defective cell.

15. Keep the level of the electrolyte about half an inch above the top of the plates by the addition, towards the end of a charge, of pure water through a funnel to the bottom of the cells.

16. *Only add water towards the end of a charge.*

Records

17. Log the specific gravity and voltage of every cell once a week in the case of installations up to 110 volts. Batteries on circuits of

from 200 volts and upwards may be divided into sections and readings on one section logged weekly, if this is approved by the E.P.S. Co.

Cells lightly worked or out of Commission

18. If cells are out of work, keep the level of the electrolyte right by the addition of distilled water and bring the cells to a fully charged condition fortnightly. If cells are to be idle for more than two months, the E.P.S. Co. should be consulted.

If cells are lightly worked, or if the specific gravities show signs of getting low, they may be charged at the maximum rate once fortnightly until the plates begin to evolve gas freely, and the charge then finished at the normal rate. This will help to keep the specific gravity of the electrolyte more uniform throughout.

Connections

19. Where cells are bolted together, keep all the connections clean and tight.

N.B.—For cells in glass containers a portable hand inspection lamp and for cells in opaque containers a special cell inspection lamp must be provided. A cell testing voltmeter and hydrometer are also necessities.

3. Care of Wireless Apparatus

As it is essential that wireless apparatus should always be kept in perfect working order, and as an operator, especially on board ship, may be out of reach of technical assistance, a few notes are given here on the care of apparatus.

Dynamos and Alternators.—Machines for generating unidirectional electric currents are generally called *dynamos*, and machines for generating alternating currents are called *alternators*. These machines are either driven by direct coupling to a steam engine or else by direct coupling to an electric motor. The most convenient way of driving an alternator is by coupling to a direct current motor which takes its current either from ship-lighting circuits or from a public electric supply. Such machines should be regularly examined and cleaned, and especially care taken about the lubrication of the bearing surfaces. These are generally now made with ball-bearings. The great point is to keep the machines free from moisture either by water or steam. For this purpose they should be frequently examined and cleaned with a dry cloth. Electric motors should be provided with a proper starting switch containing a rheostat, and also a no-load and over-load

release, so that the current cannot be put into the motor too quickly, and is cut off if the motor becomes overloaded. Particular care should be taken to see that there is no injurious sparking at the brushes. The motor commutator should be carefully cleaned, and if any sparking does take place, it should be remedied by carefully setting the brushes in the proper non-sparking position. This can be found by trial. The same care should be taken of dynamo commutators. The machine should be carefully examined when running under load to see if any sparking is taking place at the brushes, and if so, these should be carefully adjusted as to pressure and trueness of surface. The current from the machine should be led to a switchboard provided with ammeter and voltmeter so that the attendant can see at once if the machine is giving its proper voltage. In the case of alternators a speed indicator should be provided, so that the attendant may also know whether the frequency of the machine is right. Care should be taken that all connections to terminals are thoroughly good and made between clean surfaces. The insulation of the machine should be tested from time to time with a battery cell and detector to ascertain if the insulation between the coils and the iron magnets or of the machine is perfect. This test may also be made with the instrument called a megger.

Transformers and Induction Coils.—If transformers larger than 1 or 2 R.W. are employed, they are nearly always contained in iron cases and filled in with insulating oil. These cases should be perfectly watertight, and the oil should completely cover the transformer. High-tension transformers in iron boxes should have these cases well earthed, either to an earth-plate or, if on board ship, to the hull of the vessel. Induction coils should be carefully preserved from moisture, and if the coil is covered with ebonite, as is usually the case, this should be protected from the action of light by being well varnished. An induction coil intended for use with direct currents has to be provided with a current interrupter. This is generally in the form of a hammer break. This break has the advantage of simplicity, but requires some care to get the best results from it. The platinum contacts require to be trued up occasionally, but the adjusting screws should not be unnecessarily altered. For land stations, a far better form of interrupter is the mercury break filled with coal gas. For details of all these interrupters, see J. A. Fleming, *The Principles of Electric Wave Telegraphy*, chap. i.

Condensers and Leyden Jars.—Condensers for use with spark

transmitters, where great portability is required, are made of glass tubes copper plated on part of both surfaces, or, for larger plant, glass-plate condensers are employed, placed in oil included in iron boxes, or the glass plates may be dispensed with and simple metal plates placed in oil. If such oil condensers in iron boxes are employed, they should be carefully watched to see that no leakage is taking place, and inspected to see if the oil at all times entirely covers the metal plates. Any noise taking place in the interior should at once call for attention as probably indicating sparking across.

Aerials and Insulation.—Insulators upholding the antenna and also insulators by which it is led into the signalling room or cabin, should be carefully examined from time to time, and the insulation of the antenna should be tested. This can best be done by one of the "Megger" testing sets containing a small dynamo giving an electromotive force of 500 volts. One terminal of the megger can be connected to the antenna and the other to the earth, and the insulation measured. Any reduction in resistance below 100,000 ohms or so should be remedied by attention to the insulation.

Receivers and Tuning.—Receiving sets will generally be provided with a proper tuner, and in modern commercial work the reception is nearly always by ear, employing a head telephone of suitable resistance, and either a Marconi magnetic detector or a Fleming oscillation valve, or some form of crystal rectifier as a detecting device. Descriptions of these appliances and their mode of operation will be found in most books on wireless telegraphy (see J. A. Fleming, *The Principles of Electric Wave Telegraphy*, chap. vi.). If a crystal detector is used or contact rectifier, such as a "Perikon," the operator must discover by trial the pressure and contact surface which gives the best results, and this can only be done by experience and practice. In the case of the Fleming oscillation valve, the carbon or metallic filament should be brought to bright incandescence by an insulated battery, but if overheated the valve filament will be destroyed. In the case of the Marconi magnetic detector, the setting of the permanent magnets is important, and these should not be unnecessarily interfered with if adjusted by the manufacturers. All apparatus for use on board ship should, as far as possible, be kept in watertight cases, and ebonite surfaces should be well varnished to prevent loss of insulation, and all metallic contact surfaces, terminals, etc., carefully kept clean and bright.

4. Glossary of Terms used in Radiotelegraphy

Aerial or Antenna is the part of a radiotelegraphic transmitter or receiver which radiates or absorbs the electric waves. It consists of a wire or collection of wires insulated at the upper or outer end, the wires being either nearly vertical or partly vertical and partly horizontal. These wires are sustained by masts or towers. Certain descriptive adjectives are employed, such as *plain aerial* for a directly-charged aerial with two spark balls at the base, T-aerial for one which is T-shaped, *directive aerial* for one which projects its waves more in one direction than others.

Aerial Circuit. — An aerial wire or antenna possesses capacity with respect to the earth and the wires themselves possess inductance. The aerial wire, therefore, together with any coils of wire or condensers in the receiving or transmitting circuit, constitutes a circuit having capacity and inductance, and therefore has a natural electrical time-period of its own. This circuit is called the aerial circuit.

Aether.—The imponderable medium which is assumed to pervade all space, certain forms of vibration in which constitute electromagnetic waves, and therefore waves of light or radiant heat. The aether must be supposed to possess a certain absolute dielectric coefficient, and also a certain absolute magnetic permeability. The velocity of propagation of a wave through the aether is 3×10^{10} centimetres per second, and is inversely proportional to the square root of the product of the absolute dielectric coefficient and magnetic permeability of the aether.

Alternating Current.—A current which periodically changes its direction of flow, the periodic time being the interval between two maximum movements of current in the same direction.

Alternator.—A machine for the production of alternating currents.

Aperiodic.—A circuit is said to be aperiodic when it has no well-defined natural time-period of oscillation of its own.

Arrester, Earth.—A spark-gap of small spark length but large sparking surface, which is inserted near the base of a transmitting antenna, the connections to the receiving circuit being taken off either side of the spark-gap. This earth arrester obviates the necessity for any change-over switch for changing from transmission to reception, and it enables the receiving operator to keep the telephones on his head and

thus be in a position between the times of his own sparking to be called up by the corresponding station.

Atmospherics. — Disturbances produced in the aether by natural electrical discharges, such as thunderstorms. These produce false signals or sounds in the telephone which, in some cases, interfere with the reception of message-bearing signals. As a general rule, atmospherics are worse at night and greater in the summer than in the winter, and in the tropics than in temperate climates. They are also called "strays" or "X^s" or "vagrant waves."

Automatic Transmission and Reception. — A system of sending and receiving which abolishes the necessity for personal manipulation of the sending key. The messages are punched out on paper strips in such a way as to secure perfect spacing. The punched strip is then sent through an instrument called an automatic transmitter which operates a relay which in turn operates a high-tension sending key. Automatic reception is conducted by means of recorders, either photographic recorders or else a very sensitive relay operated by the antenna current, but actuating in turn a Morse inker. By this means a speed of transmission of a hundred or more words per minute can be obtained.

Autotransformer.—A coil of wire either wound on an iron core or else on an air core, which coil is divided into two parts by an electrical terminal. Such a coil is used as a transformer for increasing or diminishing electric potential or current, one section of the coil acting as the secondary circuit and the other or the whole as the primary circuit.

Battery.—A collection of voltaic cells or of Leyden jars joined up in series. In fact any collection of units multiplied and used together is called a battery.

Blower, Motor.—A useful device for creating a continuous blast of air. The usual form is a sort of fan in a box driven by an electric motor. It is used for preventing the formation of an electic arc by blowing upon the spark-balls.

Bus Bars.—A pair of conductors to which a number of electrical devices are connected in parallel, whether they be storage battery cells, dynamos, or lamps.

Buzzer.—An instrument for rapidly making and breaking an electric circuit. It consists of an electromagnet with a vibrating armature like an electric bell, or it may be a diaphragm like a telephone. The vibrations given are sufficiently rapid to cause a buzzing noise.

Buzzer, Practice Set.—A combination of a buzzer and a

signalling key arranged on a common baseboard, used for the purpose of practising Morse signalling.

Call Bell.—An arrangement giving an audible sound, used for calling up an operator to be ready to receive signals. It is generally actuated by long dash signals sent by the sending operator.

Capacity.—The quality of any conductor in virtue of which it stores up electric energy in the form of an electric charge. It is measured by the quantity of electricity in coulombs required to charge the conductor to a potential of one volt.

Characteristic Curve.—A curve so drawn that it delineates the relation between the current flowing into any electrical device and potential difference at its terminals. In the case of an ordinary metallic conductor the characteristic curve is a straight line passing through the origin of current and voltage. In the case of a crystal conductor or rectifying contact, the characteristic curve is a curved line concave upwards, and is not symmetrical on either side of the zero point or origin.

Choking Coil.—A coil of wire wound to have a large inductance.

Circuit, Closed, Oscillating.—The path in which an electric current flows is called a circuit. It may consist partly of a metallic conductor and partly of the dielectric of a condenser. If this condenser has its plates very near together, so that the lines of electrostatic force are mostly contained between the two plates, the circuit is called a closed oscillating circuit.

Circuit, Open or Radiating.—A circuit comprising in part a metallic conductor and in part a condenser, the plates or surfaces of which are very far apart, so that the lines of electrostatic force extending from one plate to the other stretch far out into the surrounding space.

Coherer.—An imperfect contact between metallic surfaces, or a mass of metallic filings which when acted upon by an alternating high frequency electromotive force improves in conductivity.

Compass, Wireless.—A name given to a form of radio-telegraphic direction-finder by which the bearings of a transmitting station can be ascertained at a receiving station.

Commutator.—Any arrangement by which the connections between circuits are changed-over, or varied, or interrupted in any way.

Condenser.—Two metallic plates or surfaces separated by a dielectric, one plate being charged positively and the other negatively, and the dielectric subjected therefore to electric

strain. Any insulated conductor in the air forms with the air and the conducting earth a condenser of a certain kind. If the dielectric between plates is air, the arrangement is called an air condenser; if oil or glass or ebonite, it is called an oil or glass-plate or ebonite-plate condenser.

Contact Rectifier.—The name given to the contact between two substances which has the property of conducting electricity better in one direction than the other. Thus a contact rectifier can be made by a contact between galena and plumbago. (See Chap. X.)

Coupling, Coefficient of.—This is generally expressed as a percentage and denotes the ratio of the mutual inductance of two circuits to the square root of the product of their separate self-inductances. The coupling is thus spoken of as a 5 or 10 per cent coupling or more. It is generally called weak when less than 5 per cent, and strong or closed when more than 10 per cent.

Crystal Detector.—A form of oscillation detector depending on the fact that certain crystals supported in a particular manner between electrodes conduct electricity better in one direction than in the opposite.

Cut-out.—Any arrangement by which an electric circuit automatically opens itself if the current exceeds a certain value. The most ordinary form of cut-out is a fusible wire, but for larger currents a magnetic cut-out is employed in which an electromagnet traversed by the current opens a switch when the current exceeds a certain fixed value.

Cymometer.—An instrument for measuring the length of electromagnetic waves and making certain other important measurements in connection with radiotelegraphy. (See Chap. VII.)

Cymoscope.—The name for any form of detector for electric waves.

Damping.—The gradual decrease in the amplitude of electric oscillations brought about in consequence of some source of dissipation of energy, either frictional resistance or else radiation.

Damping Factor.—The numerical or algebraic expression giving the ratio between the amplitude of one oscillation and the one preceding it either in the same or opposite direction.

Decrement, Logarithmic.—The logarithm of the ratio of the amplitude of one oscillation to the next oscillation in the same or opposite direction. If the two successive oscillations are in the opposite direction then the decrement is called the

decrement per semi-period, and if the two successive oscillations are those in the same direction the decrement is called the decrement per complete period.

Decremeter.—Any apparatus for measuring the logarithmic decrement.

Detector.—Any appliance which is influenced by the feeble oscillatory currents in the receiving antenna or receiving circuits of a wireless telegraph apparatus, in such fashion as to actuate another appliance which gives a visible or audible signal. It may consist of a galvanometer such as the Einthoven.

Dielectric.—Any non-conducting medium, such as air, oil, glass, ebonite, etc., which can experience electric strain when acted upon by electric force. The ratio of the electric strain or displacement per square centimetre to the electric force when multiplied by 4π gives us the dielectric constant or coefficient of the dielectric.

Dielectric Strength.—The quality of dielectric in virtue of which it can just resist a certain electric force without breaking down. It is generally expressed in kilovolts per centimetre. Thus the dielectric strength of air is about 38, which means to say that 38,000 volts will just cause a spark between two curved metal surfaces about a centimetre apart in air.

Diffraction.—The bending of waves round an obstacle. In the case of the earth the electromagnetic waves sent out from any wireless transmitter follow round the earth's curvature to a large extent. Part of this bending is due to diffraction. The degree of diffraction depends in general upon the ratio between the wave-length of the waves and the size of the obstacle. Thus diffraction is very small for light waves, but much larger relatively for long electromagnetic waves.

Direct Coupling.—When one circuit is connected with another circuit in such fashion that one part, whether coil of wire or condenser, is common to the two circuits, they are said to be directly coupled.

Discharger.—Any piece of apparatus in the circuit of a wireless telegraph transmitter between which a spark passes which can be controlled as to spark frequency and spark length.

Disc Discharger.—A form of discharger invented by Mr. Marconi and involving a rapidly rotated disc used for the production of regular trains of slightly damped waves.

Duplex Radiotelegraphy.—A system of sending and receiving in which the two operations can go on simultaneously without interference.

Earth Connection or Earths.—The connection to the upper portion of the earth's crust which is used in most systems of wireless telegraphy takes the form of a number of metal plates or wires embedded in the ground. (See Chap. IX.)

Efficiency.—Generally expressed as a percentage, and is the number which denotes the ratio between the output of any apparatus for transforming energy and the intake of energy, both being reckoned in the same units.

Electrolysis.—The chemical decomposition of a liquid or substance brought about by an electric current. The metallic or other solid surfaces by which the current enters and leaves the electrolyte are called the *electrodes*, and the ingredients into which the electrolyte is resolved are called the *ions*.

Frequency.—When any repeated change or operation takes place, such for instance as the movement of electricity backwards and forwards in a circuit or an intermittent spark between the surfaces, the number of complete waves or cycles per second is called the frequency. Thus in the case of a spark transmitter (see Chap. IX.), we have to distinguish between the spark frequency or number of sparks per second and the oscillation frequency or number of oscillations per second in each train produced by each spark. The frequency is called high or low according to a certain conventional scale. Thus an alternating current with a frequency of a hundred is called low frequency, but if the frequency is 10,000 or upwards it will be called high frequency.

Fundamental.—Any mode of oscillation can be resolved, in virtue of Fourier's theorem, into a number of simple oscillations, each of which can be represented by a simple sine curve. These constituent oscillations are called the *harmonics*, and the first of these of longest wave-length is called the *fundamental*. The wave-lengths are in the ratio of the numbers, 1, $\frac{1}{2}$, $\frac{1}{3}$, etc. When two circuits have the same period of oscillation they are said to be in tune with each other, but it is possible for one circuit to oscillate in tune with the harmonic of another circuit, in which case we may say that either the first, second, or third, etc., harmonic resonates to the fundamental of the other circuit.

High Frequency Resistance.—The resistance of a circuit is a quality of it in virtue of which it dissipates energy when a current passes through it. Owing to the fact that high frequency currents are concentrated at the surface of a conductor, the resistance of the high frequency circuit is numerically different from the resistance of the same conductor

for steady or unvarying current (see Chap. III.). The high frequency resistance may therefore be defined as the number by which the mean square value of the current in the circuit must be multiplied to obtain the power dissipated in the conductor.

Hysteresis.—A quality in virtue of which some effect produced in a body lags behind the producing cause. Thus for instance in the case of the magnetisation of iron, if the magnetising force is carried through a cycle of operations, the magnetisation at such stage lags behind the magnetising force, and we obtain what is called a hysteresis loop for the iron when these two quantities are delineated graphically. Similarly in a dielectric, the electric displacement or strain may lag behind the electric force in phase, and this happens in virtue of dielectric hysteresis.

Inductance.—The property of an electric circuit in virtue of which energy is stored up in connection with the circuit when a current flows through it. When we are considering the inductance of a single circuit, we may define this quality as the number of lines of magnetic flux which are self-linked with the circuit, these lines being due to the current flowing through the circuit. In the case of two circuits near each other, we have also to consider the mutual inductance, which may be defined either as energy stored up in connection with the circuits when unit current flows through both of them, or it may be defined as the number of lines of magnetic flux due to unit current in one circuit, which are linked with both the circuits.

Induction Coil.—A bobbin or coil having on it two wires so arranged as to have mutual inductance in consequence of which any variation in the strength of a current flowing through one wire produces an electromotive force in the other circuit. When one of these circuits is a long fine wire circuit, and the other a short thick one, then interruption of the strong current in the thick wire circuit will give rise to very high electromotive forces in the other circuit, which may be sufficient to cause a spark between the terminals of the fine wire circuit. Such a coil is called a spark coil. One of the circuits, generally the thick wire circuit, is called the primary and the other the secondary.

Inductive Coupling.—When two oscillation circuits are so situated that the wire forming part of one circuit is the primary of an induction coil, and the wire forming part of the other circuit is the secondary of the same induction coil, these circuits are said to be inductively coupled.

Interference.—In the case of wave motion, two sets of waves which proceed from one and the same source or from duplicate sources may arrive at a distant point in such a manner that the motions or changes which constitute these waves annul one another at that point. The waves are then said to interfere.

Interrupter.—Any arrangement by which a circuit is rapidly made and broken so as to start and stop an electric current. An interrupter is generally provided in the case of an induction spark-coil. Such an interrupter may be either part of the coil itself or it may be a separate piece of apparatus. For the description of numerous forms of interrupter for induction coils see *The Principles of Electric Wave Telegraphy*, J. A. Fleming, chap. i.

Ionisation.—A gas or a liquid only conducts electricity in consequence of there being present in it certain small electrically charged elements which are called ions. These ions may in some cases be electrons, which are the small invisible units of electricity, or the ions may be chemical atoms which have gained or lost electrons and therefore become positively or negatively charged. In this case they are called atomic ions. Again, an atomic ion or an electron may gather round it a number of uncharged chemical atoms, or else a group of chemical atoms forming a molecule may gain or lose electrons. In either case it becomes charged electrically. In all cases these small charged elements are called ions, and the process of producing them either in a liquid or in a gas is called ionisation.

Jigger.—A term employed to denote a form of induction coil or oscillation transformer used for inductively transforming high frequency currents. If such a two-coil or single-coil oscillation transformer is used in a receiving circuit it is commonly called a receiving jigger, and if in a transmitting circuit it is called a sending jigger.

Key.—Any device for interrupting an electric circuit either by hand or through the intermediation of a relay, in which latter case the key is called a relay key.

Leyden Jar.—A form of condenser consisting of a glass bottle or tube copper or silver plated both inside and outside up to a certain height and with suitable connections to the two metallic surfaces.

Loop of Potential.—When electric oscillations are set up in a wire such as an antenna wire, there are some places where the amplitude of potential variation is greatest, and

others where it is least or zero. The former are called loops of potential and the latter are called nodes.

Magnetic Detector.—A form of detector for electric waves in which the variation of magnetisation of iron by electric oscillations is utilised to make an audible signal by means of a telephone. In the form invented by Mr. Marconi there is an endless travelling band of soft iron wires which passes in front of a pair of horse-shoe magnets, and also through a coil having wound on it a wire through which the oscillations to be detected pass.

Micrometer.—Any instrument for measuring a very small interval of space. For example, a micrometer spark-gap comprising two spark balls, one of which can be moved over a very small distance by a screw, so as to adjust the spark-gap length for measuring.

Microphone.—The name generally given to some arrangement in which the sound of the speaking voice is caused to vary a resistance. In the commonest form of microphone, called a carbon microphone, the variations in air pressure due to the speaking voice are made to act on a diaphragm which compresses more or less some carbon granules, and this varies the resistance in the circuit in which these granules are included.

Morse Inker.—An instrument for recording Morse code signals in the form of dots and dashes on a travelling paper band.

Non-Inductive Circuit.—A circuit which possesses very small or negligible inductance. Such an inductance can be made by doubling a single length of wire and winding it on a bobbin, so that a current flowing through it makes as many turns one way as the other, and there is little or no self-linked magnetic field.

Oscillations.—Alternating currents of very high frequency are called electrical oscillations. If the amplitude is constant they are called sustained or persistent or undamped oscillations; but if the amplitude dies away, they are called damped oscillations. If the oscillations take place with the natural frequency of the circuit and without any external impressed electromotive force, they are called free oscillations, but if they take place under the action of an external E.M.F., the frequency of which is not in accord with the natural frequency of the circuit, they are called forced oscillations.

Periodic Time.—The time of one complete oscillation in a circuit, whether free or forced.

Phase. — When an oscillation goes through a complete cycle within a certain periodic time, its condition at any one moment may be expressed by stating the fraction of the periodic time which has elapsed since the quantity last passed through its zero value, and this is called its phase. It is generally expressed in terms of angular measurement, *i.e.* at so many degrees, taking the complete period to be 360 degrees. Two periodic quantities may therefore be said to agree or differ in phase.

Plain Aerial.—A single or multiple wire antenna which is self-charged, thus forming at the same time the radiator and a condenser. It has two spark balls inserted near the base, and was used by Mr. Marconi in his earliest experiments on radiotelegraphy.

Potentiometer.—A wire or series of coils of wire having an E.M.F. applied at the ends by means of a battery or cell and a sliding contact provided on the wire or the coils so as to enable us to tap off any required fraction of the potential of that battery or cell. Such an arrangement is called a potentiometer.

Quenched Spark.—A form of electric spark which, owing to the nature of the electrodes, is rapidly extinguished after allowing one or two oscillations to pass.

Quenched Spark Transmitter.—A radiotelegraphic transmitter employing a quenched spark.

Radiation.—The transmission of energy through space in the form of electromagnetic waves. If these waves are very short, namely, a small fraction of an inch, the radiation is called heat or light; if the waves are much longer, the wavelength being measured in feet or miles, it is called electromagnetic radiation.

Radiation Resistance.—The quality of an antenna in virtue of which radiation of energy takes place. It may be measured by the quotient of the rate of radiation reckoned in watts by the mean square value of the current in the antenna. (See Chap. VIII., p. 193.)

Rectifier.—Any apparatus for converting alternating currents or electric oscillations into a continuous current.

Refraction. — The bending of a ray of electromagnetic radiation of any kind which takes place when that radiation passes from a medium having one dielectric constant to another medium having a different dielectric constant.

Relay.—Any apparatus by means of which changes in a small current can be made to create corresponding changes in a large current furnished by a local battery.

Resonance.—When electric oscillations of a certain frequency are taking place in one circuit, and if another circuit is brought near to the first, the natural frequency of the second being capable of being changed by altering its capacity or inductance, then if the natural frequency is so changed, the current in the secondary circuit will alter, and if the natural frequency of the secondary circuit is adjusted to be equal to that of the primary circuit, then the current in the secondary circuit will come to a maximum. This action by which oscillations in one circuit produce oscillations of a maximum value in another circuit tuned to the same frequency is called the resonance. The term is derived from its analogy in acoustic phenomena.

Root-Mean-Square Value.—A periodic current which passes through a cycle of values has some one particular value called the instantaneous value at any moment. If we take the mean of the squares of the instantaneous values at equidistant intervals of time throughout the period, the square root of the mean of these values is called the root-mean-square value. In a current of simple sine form the R.M.S. value is equal to the maximum value multiplied by 0.707.

Shock-Excitation.—A name given to the method of exciting oscillations by means of a quenched spark (see *Quenched Spark*).

Specific Inductive Capacity.—The quality of a dielectric which determines the ratio of the electric strain to the electric force producing it. Otherwise specific inductive capacity may be defined as the ratio of the capacity of a condenser having a dielectric of this particular material to the capacity of the same condenser if the dielectric is replaced by air.

Syntony and Syntonisation.—Two circuits are said to be in syntony when they have the same natural period of electrical oscillation.

Train of Waves.—A group of oscillations is sent out from an aerial at each discharge of the condenser, if the transmitter is arranged on the spark system. If a number of trains of oscillations are sent out at equidistant intervals, the number per second is called the train frequency or group frequency, and this must be distinguished from the oscillation frequency.

Transformer.—An arrangement consisting of two coils of wire wound either on an air core or an iron core. When an alternating current or oscillation is sent through one of these wires, it creates another alternating current in the second circuit, and this secondary current may have a different strength and E.M.F. from the primary. We can therefore

increase the potential by reducing current and *vice versa*, and a transformer is called a step-up or step-down transformer according as it raises or lowers voltage.

Tuner.—A piece of apparatus providing in a compact form all the arrangements necessary for the receiving circuits of a wireless telegraph receiver. It comprises certain condensers, inductance coils, and jiggers, the capacity, inductance, and coupling of which can be varied so as to bring the various circuits into resonance with each other.

Undamped Waves.—A train of persistent waves the amplitude of which remains always the same.

Valve, Vacuum.—A form of detector for electric oscillations invented by the author. It consists of an electric glow-lamp, either metallic filament or carbon filament, having a cylinder adjacent to or surrounding the filament but not touching it. This is carried on a wire sealed through the bulb and connected to a third terminal. When the filament is rendered incandescent by a continuous current the space between the filament and the metal plate possesses a unilateral conductivity, and negative electricity can pass from the filament to the plate and not in the opposite direction. It is used as a rectifier of electric oscillations to rectify trains of oscillations into gushes of electricity in the same direction, and render them audible on a telephone.

Wave-length.—The shortest distance between two parts of the medium executing periodic motions or changes which come to their maximum value or zero value at the same instant. In the case of water waves, the wave-length is the shortest distance from crest to crest, measured transversely, to the line of wave elevation.

Wavemeter.—Any instrument for measuring the length or frequency of electromagnetic waves (see *Cymometer*).

Waves and Wave Motion.—When the different adjacent parts of a medium undergo any periodic change and not simultaneously but successively, one after the other, in such fashion that each part executes a periodic change or motion, this is called a wave motion. We have instances of the wave motion in sound waves, light waves, and electric waves.

CHAPTER XII

MATHEMATICAL TABLES, TABLES OF VARIOUS CONSTANTS, AND TABLES OF WIRE SIZES

1. Table of Squares, Cubes, Square Roots, and Cube Roots of Numerals from 1 to 1000.
2. Table of Ordinary Logarithms from 1 to 1000.
3. Table of Napierian Logarithms from 1 to 100.
4. Table of Natural Sines, Cosines, and Tangents.
5. Table of Circular and Hyperbolic Functions of Radians.
6. Table of Wire Gauges.
7. Table of Sizes, Weights, and Resistances of Copper Wires.
8. Table of Sizes, Weights, and Resistances of Aluminium Wires.
9. Table of Various Data with regard to Copper and Aluminium.
10. Volume Resistivities of Pure Metals.
11. Volume Resistivities of some Alloys.
12. Table of Dielectric Constants of Solids.
13. Table of Dielectric Constants of Liquids.
14. Table of Dielectric Constants of Gases.

1. TABLE OF SQUARES, CUBES, SQUARE ROOTS, AND CUBE ROOTS OF NUMERALS FROM 1 TO 1000.

No.	Square.	Cube.	Square Root.	Cube Root.
1	1	1	1.000000	1.000000
2	4	8	1.414214	1.259921
3	9	27	1.732051	1.442250
4	16	64	2.000000	1.587401
5	25	125	2.236068	1.709976
6	36	216	2.449490	1.817121
7	49	343	2.645751	1.912931
8	64	512	2.828427	2.000000
9	81	729	3.000000	2.080084
10	100	1000	3.162278	2.154435

No.	Square.	Cube.	Square Root.	Cube Root.
11	121	1331	3.316625	2.223980
12	144	1728	3.464102	2.289429
13	169	2197	3.605551	2.351335
14	196	2744	3.741657	2.410142
15	225	3375	3.872983	2.466212
16	256	4096	4.000000	2.519842
17	289	4913	4.123106	2.571282
18	324	5832	4.242641	2.620741
19	361	6859	4.358899	2.668402
20	400	8000	4.472136	2.714418
21	441	9261	4.582576	2.758924
22	484	10648	4.690416	2.802039
23	529	12167	4.795832	2.843867
24	576	13824	4.898979	2.884499
25	625	15625	5.000000	2.924018
26	676	17576	5.099020	2.962496
27	729	19683	5.196152	3.000000
28	784	21952	5.291503	3.036589
29	841	24389	5.385165	3.072317
30	900	27000	5.477226	3.107232
31	961	29791	5.567764	3.141381
32	1024	32768	5.656854	3.174802
33	1089	35937	5.744563	3.207534
34	1156	39304	5.830952	3.239612
35	1225	42875	5.916080	3.271066
36	1296	46656	6.000000	3.301927
37	1369	50653	6.082763	3.332222
38	1444	54872	6.164414	3.361975
39	1521	59319	6.244998	3.391211
40	1600	64000	6.324555	3.419952
41	1681	68921	6.403124	3.448217
42	1764	74088	6.480741	3.476027
43	1849	79507	6.557439	3.503398
44	1936	85184	6.633250	3.530348
45	2025	91125	6.708204	3.556893
46	2116	97336	6.782330	3.583048
47	2209	103823	6.855655	3.608826
48	2304	110592	6.928203	3.634241
49	2401	117649	7.000000	3.659306
50	2500	125000	7.071068	3.684031

No.	Square.	Cube.	Square Root.	Cube Root.
51	2601	132651	7.141428	3.708430
52	2704	140608	7.211103	3.732511
53	2809	148877	7.280110	3.756286
54	2916	157464	7.348469	3.779763
55	3025	166375	7.416198	3.802953
56	3136	175616	7.483315	3.825862
57	3249	185193	7.549834	3.848501
58	3364	195112	7.615773	3.870877
59	3481	205379	7.681146	3.892996
60	3600	216000	7.745967	3.914868
61	3721	226981	7.810250	3.936497
62	3844	238328	7.874008	3.957892
63	3969	250047	7.937254	3.979057
64	4096	262144	8.000000	4.000000
65	4225	274625	8.062258	4.020726
66	4356	287496	8.124038	4.041240
67	4489	300763	8.185353	4.061548
68	4624	314432	8.246211	4.081655
69	4761	328509	8.306624	4.101566
70	4900	343000	8.366600	4.121285
71	5041	357911	8.426150	4.140818
72	5184	373248	8.485281	4.160168
73	5329	389017	8.544004	4.179339
74	5476	405224	8.602325	4.198336
75	5625	421875	8.660254	4.217163
76	5776	438976	8.717798	4.235824
77	5929	456533	8.774964	4.254321
78	6084	474552	8.831761	4.272659
79	6241	493039	8.888194	4.290840
80	6400	512000	8.944272	4.308870
81	6561	531441	9.000000	4.326749
82	6724	551368	9.055385	4.344481
83	6889	571787	9.110434	4.362071
84	7056	592704	9.165151	4.379519
85	7225	614125	9.219544	4.396830
86	7396	636056	9.273618	4.414005
87	7569	658503	9.327379	4.431048
88	7744	681472	9.380832	4.447960
89	7921	704969	9.433981	4.464745
90	8100	729000	9.486833	4.481405

No.	Square.	Cube.	Square Root.	Cube Root.
91	8281	753571	9.539392	4.497941
92	8464	778688	9.591663	4.514357
93	8649	804357	9.643651	4.530655
94	8836	830584	9.695360	4.546836
95	9025	857375	9.746794	4.562903
96	9216	884736	9.797959	4.578857
97	9409	912673	9.848858	4.594701
98	9604	941192	9.899495	4.610436
99	9801	970299	9.949874	4.626065
100	10000	1000000	10.000000	4.641589
101	10201	1030301	10.049876	4.657010
102	10404	1061208	10.099505	4.672329
103	10609	1092727	10.148892	4.687548
104	10816	1124864	10.198039	4.702669
105	11025	1157625	10.246951	4.717694
106	11236	1191016	10.295630	4.732624
107	11449	1225043	10.344080	4.747459
108	11664	1259712	10.392305	4.762203
109	11881	1295029	10.440306	4.776856
110	12100	1331000	10.488088	4.791420
111	12321	1367631	10.535654	4.805896
112	12544	1404928	10 583005	4.820284
113	12769	1442897	10.630146	4.834588
114	12996	1481544	10.677078	4.848808
115	13225	1520875	10.723805	4.862944
116	13456	1560896	10.770330	4.876999
117	13689	1601613	10.816654	4.890973
118	13924	1643032	10.862780	4.904868
119	14161	1685159	10.908712	4.918685
120	14400	1728000	10.954451	4.932424
121	14641	1771561	11.000000	4.946087
122	14884	1815848	11.045361	4.959676
123	15129	1860867	11.090536	4.973190
124	15376	1906624	11.135529	4.986631
125	15625	1953125	11.180340	5.000000
126	15876	2000376	11.224972	5.013298
127	16129	2048383	11.269428	5.026526
128	16384	2097152	11.313708	5.039684
129	16641	2146689	11.357817	5.052774
130	16900	2197000	11.401754	5.065797

No.	Square.	Cube.	Square Root.	Cube Root.
131	17161	2248091	11.445523	5.078753
132	17424	2299968	11.489125	5.091643
133	17689	2352637	11.532563	5.104469
134	17956	2406104	11.575837	5.117230
135	18225	2460375	11.618950	5.129928
136	18496	2515456	11.661904	5.142563
137	18769	2571353	11.704700	5.155137
138	19044	2628072	11.747344	5.167649
139	19321	2685619	11.789826	5.180101
140	19600	2744000	11.832160	5.192494
141	19881	2803221	11.874342	5.204828
142	20164	2863288	11.916375	5.217103
143	20449	2924207	11.958261	5.229321
144	20736	2985984	12.000000	5.241483
145	21025	3048625	12.041595	5.253588
146	21316	3112136	12.083046	5.265637
147	21609	3176523	12.124356	5.277632
148	21904	3241792	12.165525	5.289572
149	22201	3307949	12.206556	5.301459
150	22500	3375000	12.247449	5.313293
151	22801	3442951	12.288206	5.335074
152	23104	3511808	12.328828	5.336803
153	23409	3581577	12.369317	5.348481
154	23716	3652264	12.409674	5.360108
155	24025	3723875	12.449900	5.371685
156	24336	3796416	12.489996	5.383213
157	24649	3869893	12.529964	5.394691
158	24964	3944312	12.569805	5.406120
159	25281	4019679	12.609520	5.417501
160	25600	4096000	12.649111	5.428835
161	25921	4173281	12.688577	5.440122
162	26244	4251528	12.727922	5.451362
163	26569	4330747	12.767145	5.462556
164	26896	4410944	12.806248	5.473704
165	27225	4492125	12.845233	5.484807
166	27556	4574296	12.884099	5.495865
167	27889	4657463	12.922848	5.506878
168	28224	4741632	12.961481	5.517848
169	28561	4826809	13.000000	5.528775
170	28900	4913000	13.038405	5.539658

No.	Square.	Cube.	Square Root.	Cube Root.
171	29241	5000211	13.076697	5.550499
172	29584	5088448	13.114877	5.561298
173	29929	5177717	13.152946	5.572055
174	30276	5268024	13.190906	5.582770
175	30625	5359375	13.228757	5.593445
176	30976	5451776	13.266499	5.604079
177	31329	5545233	13.304135	5.614672
178	31684	5639752	13.341664	5.625226
179	32041	5735339	13.379088	5.635741
180	32400	5832000	13.416408	5.646216
181	32761	5929741	13.453624	5.656653
182	33124	6028568	13.490738	5.667051
183	33489	6128487	13.527749	5.677411
184	33856	6229504	13.564660	5.687734
185	34225	6331625	13.601470	5.698019
186	34596	6434856	13.638182	5.708267
187	34969	6539203	13.674794	5.718479
188	35344	6644672	13.711309	5.728654
189	35721	6751269	13.747727	5.738794
190	36100	6859000	13.784049	5.748897
191	36481	6967871	13.820275	5.758965
192	36864	7077888	13.856406	5.768998
193	37249	7189057	13.892444	5.778997
194	37636	7301384	13.928388	5.788960
195	38025	7414875	13.964240	5.798890
196	38416	7529536	14.000000	5.808786
197	38809	7645373	14.035669	5.818648
198	39204	7762392	14.071247	5.828477
199	39601	7880599	14.106736	5.838272
200	40000	8000000	14.142136	5.848035
201	40401	8120601	14.177447	5.857766
202	40804	8242408	14.212670	5.867464
203	41209	8365427	14.247807	5.877130
204	41616	8489664	14.282857	5.886765
205	42025	8615125	14.317821	5.896368
206	42436	8741816	14.352700	5.905941
207	42849	8869743	14.387495	5.915482
208	43264	8998912	14.422205	5.924992
209	43681	9129329	14.456832	5.934473
210	44100	9261000	14.491377	5.943922

No.	Square.	Cube.	Square Root.	Cube Root.
211	44521	9393931	14.525839	5.953342
212	44944	9528128	14.560220	5.962731
213	45369	9663597	14.594520	5.972091
214	45796	9800344	14.628739	5.981426
215	46225	9938375	14.662878	5.990727
216	46656	10077696	14.696939	6.000000
217	47089	10218313	14.730920	6.009244
218	47524	10360232	14.764823	6.018463
219	47961	10503459	14.798649	6.027650
220	48400	10648000	14.832397	6.036811
221	48841	10793861	14.866069	6.045943
222	49284	10941048	14.899664	6.055048
223	49729	11089567	14.933185	6.064126
224	50176	11239424	14.966630	6.073178
225	50625	11390625	15.000000	6.082201
226	51076	11543176	15.033296	6.091199
227	51529	11697083	15.066519	6.100170
228	51984	11852352	15.099669	6.109115
229	52441	12008989	15.132746	6.118033
230	52900	12167000	15.165751	6.126925
231	53361	12326391	15.198684	6.135792
232	53824	12487168	15.231546	6.144634
233	54289	12649337	15.264338	6 153449
234	54756	12812904	15.297059	6.162240
235	55225	12977875	15.329710	6.171006
236	55696	13144256	15.362292	6.179747
237	56169	13312053	15 394804	6.188463
238	56644	13481272	15.427249	6.197154
239	57121	13651919	15.459625	6.205822
240	57600	13824000	15.491933	6.214465
241	58081	13997521	15.524175	6.223084
242	58564	14172488	15.556349	6.231680
243	59049	14348907	15.588457	6.240251
244	59536	14526784	15.620499	6.248800
245	60025	14706125	15.652476	6.257325
246	60516	14886936	15.684387	6.265827
247	61009	15069223	15.716234	6.274305
248	61504	15252992	15.748016	6.282761
249	62001	15438249	15.779734	6.291195
250	62500	15625000	15.811388	6.299605

No.	Square.	Cube.	Square Root.	Cube Root.
251	63001	15813251	15.842980	6.307994
252	63504	16003008	15.874508	6.316360
253	64009	16194277	15.905974	6.324704
254	64516	16387064	15.937378	6.333026
255	65025	16581375	15.968719	6.341326
256	65536	16777216	16.000000	6.349604
257	66049	16974593	16.031220	6.357861
258	66564	17173512	16.062378	6.366097
259	67081	17373979	16.093477	6.374311
260	67600	17576000	16.124516	6.382504
261	68121	17779581	16.155494	6.390676
262	68644	17984728	16.186414	6.398828
263	69169	18191447	16.217275	6.406958
264	69696	18399744	16.248077	6.415069
265	70225	18609625	16.278821	6.423158
266	70756	18821096	16.309506	6.431228
267	71289	19034163	16.340135	6.439277
268	71824	19248832	16.370706	6.447306
269	72361	19465109	16.401220	6.455315
270	72900	19683000	16.431677	6.463304
271	73441	19902511	16.462078	6.471274
272	73984	20123648	16.492423	6.479224
273	74529	20346417	16.522712	6.487154
274	75076	20570824	16.552945	6.495065
275	75625	20796875	16.583124	6.502957
276	76176	21024576	16.613248	6.510830
277	76729	21253933	16.643317	6.518684
278	77284	21484952	16.673332	6.526519
279	77841	21717639	16.703293	6.534335
280	78400	21952000	16.733201	6.542133
281	78961	22188041	16.763055	6.549911
282	79524	22425768	16.792856	6.557672
283	80089	22665187	16.822604	6.565414
284	80656	22906304	16.852300	6.573139
285	81225	23149125	16.881943	6.580844
286	81796	23393656	16.911535	6.588532
287	82369	23639903	16.941074	6.596202
288	82944	23887872	16.970563	6.603854
289	83521	24137569	17.000000	6.611489
290	84100	24389000	17.029386	6.619106

No.	Square.	Cube.	Square Root.	Cube Root.
291	84681	24642171	17.058722	6.626705
292	85264	24897088	17.088008	6.634287
293	85849	25153757	17.117243	6 641852
294	86436	25412184	17.146428	6.649400
295	87025	25672375	17.175564	6.656930
296	87616	25934336	17.204651	6.664444
297	88209	26198073	17.233688	6.671940
298	88804	26463592	17.262676	6.679420
299	89401	26730899	17.291617	6.686883
300	90000	27000000	17.320508	6.694329
301	90601	27270901	17.349352	6.701759
302	91204	27543608	17.378147	6.709173
303	91809	27818127	17.406895	6.716570
304	92416	28094464	17.435596	6.723951
305	93025	28372625	17.464249	6.731316
306	93636	28652616	17.492856	6.738664
307	94249	28934443	17.521416	6.745997
308	94864	29218112	17.549929	6.753313
309	95481	29503629	17.578396	6.760614
310	96100	29791000	17.606817	6.767899
311	96721	30080231	17.635192	6.775169
312	97344	30371328	17.663522	6.782423
313	97969	30664297	17.691806	6.789661
314	98596	30959144	17.720045	6.796884
315	99225	31255875	17.748239	6.804092
316	99856	31554496	17.776389	6.811285
317	100489	31855013	17.804494	6.818462
318	101124	32157432	17.832555	6.825624
319	101761	32461759	17.860571	6.832771
320	102400	32768000	17.888544	6.839904
321	103041	33076161	17.916473	6.847021
322	103684	33386248	17.944358	6.854124
323	104329	33698267	17.972201	6.861212
324	104976	34012224	18.000000	6.868285
325	105625	34328125	18.027756	6.875344
326	106276	34645976	18.055470	6.882389
327	106929	34965783	18.083141	6.889419
328	107584	35287552	18.110770	6.896435
329	108241	35611289	18.138357	6.903436
330	108900	35937000	18.165902	6.910423

No.	Square.	Cube.	Square Root.	Cube Root.
331	109561	36264691	18.193405	6.917396
332	110224	36594368	18.220867	6.924356
333	110889	36926037	18.248288	6.931301
334	111556	37259704	18.275667	6.938232
335	112225	37595375	18.303005	6.945150
336	112896	37933056	18.330303	6.952053
337	113569	38272753	18.357560	6.958943
338	114244	38614472	18.384776	6.965820
339	114921	38958219	18.411953	6.972683
340	115600	39304000	18.439089	6.979532
341	116281	39651821	18.466185	6.986368
342	116964	40001688	18.493242	6.993491
343	117649	40353607	18.520259	7.000000
344	118336	40707584	18.547237	7.006796
345	119025	41063625	18.574176	7.013579
346	119716	41421736	18.601075	7.020349
347	120409	41781923	18.627936	7.027106
348	121104	42144192	18.654758	7.033850
349	121801	42508549	18.681542	7.040581
350	122500	42875000	18.708287	7.047299
351	123201	43243551	18.734994	7.054004
352	123904	43614208	18.761663	7.060697
353	124609	43986977	18.788294	7.067377
354	125316	44361864	18.814888	7.074044
355	126025	44738875	18.841444	7.080699
356	126736	45118016	18.867962	7.087341
357	127449	45499293	18.894444	7.093971
358	128164	45882712	18.920888	7.100588
359	128881	46268279	18.947295	7.107194
360	129600	46656000	18.973666	7.113787
361	130321	47045881	19.000000	7.120367
362	131044	47437928	19.026298	7.126936
363	131769	47832147	19.052559	7.133492
364	132496	48228544	19.078784	7.140037
365	133225	48627125	19.104973	7.146569
366	133956	49027896	19.131127	7.153090
367	134689	49430863	19.157244	7.159599
368	135424	49836032	19.183326	7.166086
369	136161	50243409	19.209373	7.172581
370	136900	50653000	19.235384	7.179054

No.	Square.	Cube.	Square Root.	Cube Root.
371	137641	51064811	19.261360	7.185516
372	138384	51478848	19.287302	7.191966
373	139129	51895117	19.313208	7.198405
374	139876	52313624	19.339080	7.204832
375	140625	52734375	19.364917	7.211248
376	141376	53157376	19.390719	7.217652
377	142129	53582633	19.416488	7.224045
378	142884	54010152	19.442222	7.230427
379	143641	54439939	19.467922	7.236797
380	144400	54872000	19.493589	7.243156
381	145161	55306341	19.519221	7.249504
382	145924	55742968	19.544820	7.255841
383	146689	56181887	19.570386	7.262167
384	147456	56623104	19.595918	7.268482
385	148225	57066625	19.621417	7.274786
386	148996	57512456	19.646883	7.281079
387	149769	57960603	19.672316	7.287362
388	150544	58411072	19.697716	7.293633
389	151321	58863869	19.723083	7.299894
390	152100	59319000	19.748418	7.306144
391	152881	59776471	19.773720	7.312383
392	153664	60236288	19.798990	7.318611
393	154449	60698457	19.824228	7.324829
394	155236	61162984	19.849433	7.331037
395	156025	61629875	19.874607	7.337234
396	156816	62099136	19.899749	7.343420
397	157609	62570773	19.924859	7.349597
398	158404	63044792	19.949937	7.355762
399	159201	63521199	19.974984	7.361918
400	160000	64000000	20.000000	7.368063
401	160801	64481201	20.024984	7.374198
402	161604	64964808	20.049938	7.380323
403	162409	65450827	20.074860	7.386437
404	163216	65939264	20.099751	7.392542
405	164025	66430125	20.124612	7.398636
406	164836	66923416	20.149442	7.404721
407	165649	67419143	20.174241	7.410795
408	166464	67917312	20.199010	7.416859
409	167281	68417929	20.223748	7.422914
410	168100	68921000	20.248457	7.428959

No.	Square.	Cube.	Square Root.	Cube Root.
411	168921	69426531	20.273135	7.434994
412	169744	69934528	20.297783	7.441019
413	170569	70444997	20.322401	7.447034
414	171396	70957944	20.346990	7.453040
415	172225	71473375	20.371549	7.459036
416	173056	71991296	20.396078	7.465022
417	173889	72511713	20.420578	7.470999
418	174724	73034632	20.445048	7.476966
419	175561	73560059	20.469490	7.482924
420	176400	74088000	20.493902	7.488872
421	177241	74618461	20.518285	7.494811
422	178084	75151448	20.542639	7.500741
423	178929	75686967	20.566964	7.566964
424	179776	76225024	20.591260	7.512571
425	180625	76765625	20.615528	7.518473
426	181476	77308776	20.639767	7.524365
427	182329	77854483	20.663978	7.530248
428	183184	78402752	20.688161	7.536122
429	184041	78953589	20.712315	7.541987
430	184900	79507000	20.736441	7.547842
431	185761	80062991	20.760540	7.553689
432	186624	80621568	20.784610	7.559526
433	187489	81182737	20.808652	7.565355
434	188356	81746504	20.832667	7.571174
435	189225	82312875	20.856654	7.576985
436	190096	82881856	20.880613	7.582786
437	190969	83453453	20.904545	7.588579
438	191844	84027672	20.928450	7.594363
439	192721	84604519	20.952327	7.600138
440	193600	85184000	20.976177	7.605905
441	194481	85766121	21.000000	7.611663
442	195364	86350888	21.023796	7.617412
443	196249	86938307	21.047656	7.623152
444	197136	87528384	21.071308	7.628884
445	198025	88121125	21.095023	7.634607
446	198916	88716536	21.118712	7.640321
447	199809	89314623	21.142375	7.646027
448	200704	89915392	21.166011	7.651725
449	201601	90518849	21.189620	7.657414
450	202500	91125000	21.213203	7.663094

No.	Square.	Cube.	Square Root.	Cube Root.
451	203401	91733851	21.236761	7.668766
452	204304	92345408	21.260292	7.674430
453	205209	92959677	21.283797	7.680086
454	206116	93576664	21.307276	7.685733
455	207025	94196375	21.330729	7.691372
456	207936	94818816	21.354157	7.697002
457	208849	95443993	21.377558	7.702625
458	209764	96071912	21.400935	7.708239
459	210681	96702579	21.424285	7.713845
460	211600	97336000	21.447611	7.719443
461	212521	97972181	21.470911	7.725032
462	213444	98611128	21.494185	7.730614
463	214369	99252847	21.517435	7.736188
464	215296	99897344	21.540659	7.741753
465	216225	100544625	21.563859	7.747311
466	217156	101194696	21.587033	7.752861
467	218089	101847563	21.610183	7.758402
468	219024	102503232	21.633308	7.763936
469	219961	103161709	21.656408	7.769462
470	220900	103823000	21.679483	7.774980
471	221841	104487111	21.702534	7.780490
472	222784	105154048	21.725561	7.785993
473	223729	105823817	21.748563	7.791487
474	224676	106496424	21.771541	7.796974
475	225625	107171875	21.794495	7.802454
476	226576	107850176	21.817424	7.807925
477	227529	108531333	21.840330	7.813389
478	228484	109215352	21.863211	7.818846
479	229441	109902239	21.886069	7.824294
480	230400	110592000	21.908902	7.829735
481	231361	111284641	21.931712	7.835169
482	232324	111980168	21.954498	7.840595
483	233289	112678587	21.977261	7.846013
484	234256	113379904	22.000000	7.851424
485	235225	114084125	22.022716	7.856828
486	236196	114791256	22.045408	7.862224
487	237169	115501303	22.068077	7.867613
488	238144	116214272	22.090722	7.872994
489	239121	116930169	22.113344	7.878368
490	240100	117649000	22.135944	7.883735

No.	Square.	Cube.	Square Root.	Cube Root.
491	241081	118370771	22.158520	7.889095
492	242064	119095488	22.181073	7.894447
493	243049	119823157	22.203603	7.899792
494	244036	120553784	22.226111	7.905129
495	245025	121287375	22.248596	7.910460
496	246016	122023936	22.271058	7.915783
497	247009	122763473	22.293497	7.921099
498	248004	123505992	22.315914	7.926408
499	249001	124251499	22.338308	7.931710
500	250000	125000000	22.360680	7.937005
501	251001	125751501	22.383029	7.942293
502	252004	126506008	22.405357	7.947574
503	253009	127263527	22.427662	7.952848
504	254016	128024064	22.449944	7.958114
505	255025	128787625	22.472205	7.963374
506	256036	129554216	22.494444	7.968627
507	257049	130323843	22.516661	7.973873
508	258064	131096551	22.538855	7.979112
509	259081	131872229	22.561028	7.984344
510	260100	132651000	22.583180	7.989570
511	261121	133432831	22.605309	7.994788
512	262144	134217728	22.627417	8.000000
513	263169	135005697	22.649503	8.005205
514	264196	135796744	22.671568	8.010403
515	265225	136590875	22.693611	8.015595
516	266256	137388086	22.715633	8.020779
517	267289	138188413	22.737634	8.025957
518	268324	138991832	22.759613	8.031129
519	269361	139798359	22.781572	8.036293
520	270400	140608000	22.803509	8.041451
521	271441	141420761	22.825424	8.046603
522	272484	142236648	22.847319	8.051748
523	273529	143055667	22.869193	8.056886
524	274576	143877824	22.891046	8.062018
525	275625	144703125	22.912879	8.067143
526	276676	145531576	22.934690	8.072262
527	277729	146363183	22.956481	8.077374
528	278784	147197952	22.978251	8.082480
529	279841	148035889	23.000000	8.087579
530	280900	148877000	23.021729	8.092672

No.	Square.	Cube.	Square Root.	Cube Root.
531	281961	149721291	23.043437	8.097759
532	283024	150568768	23.065125	8.102839
533	284089	151419437	23.086793	8.107913
534	285156	152273304	23.108440	8.112980
535	286225	153130375	23.130067	8.118041
536	287296	153990656	23.151674	8.123096
537	288369	154854153	23.173261	8.128145
538	289444	155720872	23.194827	8.133187
539	290521	156590819	23.216374	8.138223
540	291600	157464000	23.237900	8.143253
541	292681	158340421	23.259407	8.148276
542	293764	159220088	23.280894	8.153294
543	294849	160103007	23.302360	8.158305
544	295936	160989184	23.323808	8.163310
545	297025	161878625	23.345235	8.168309
546	298116	162771336	23.366643	8.173302
547	299209	163667323	23.288031	8.178289
548	300304	164566592	23.409400	8.183269
549	301401	165469149	23.430749	8.188244
550	302500	166375000	23.452079	8.193213
551	303601	167284151	23.473389	8.198175
552	304704	168196608	23.494680	8.203132
553	305809	169112377	23.515952	8.208082
554	306916	170031464	23.537205	8.213027
555	308025	170953875	23.558438	8.217966
556	309136	171879616	23.579652	8.222898
557	310249	172808693	23.600847	8.227825
558	311364	173741112	23.622024	8.232746
559	312481	174676879	23.643181	8.237661
560	313600	175616000	23.664319	8.242571
561	314721	176558481	23.685439	8.247474
562	315844	177504328	23.706539	8.252371
563	316969	178453547	23.727621	8.257263
564	318096	179406144	23.748684	8.262149
565	319225	180362125	23.769729	8.267029
566	320356	181321496	23.790755	8.271904
567	321489	182284263	23.811762	8.276773
568	322624	183250432	23.832751	8.281635
569	323761	184220009	23.853721	8.286493
570	324900	185193000	23.874673	8.291344

No.	Square.	Cube.	Square Root.	Cube Root.
571	326041	186169411	23.895606	8.296190
572	327184	187149248	23.916522	8.301030
573	328329	188132517	23.937418	8.305865
574	329476	189119224	23.958297	8.310694
575	330625	190109375	23.979158	8.315517
576	331776	191102976	24.000000	8.320335
577	332929	192100033	24.020824	8.325147
578	334084	193100552	24.041631	8.329954
579	335241	194104539	24.062419	8.334755
580	336400	195112000	24.083189	8.339551
581	337561	196122941	24.103942	8.344341
582	338724	197137368	24.124676	8.349126
583	339889	198155287	24.145393	8.353905
584	341056	199176704	24.166092	8.358678
585	342225	200201625	24.186773	8.363447
586	343396	201230056	24.207437	8.368209
587	344569	202262003	24.228083	8.372967
588	345744	203297472	24.248711	8.377719
589	346921	204336469	24.269322	8.382465
590	348100	205379000	24.289916	8.387206
591	349281	206425071	24.310492	8.391942
592	350464	207474688	24.331050	8.396673
593	351649	208527857	24.351591	8.401398
594	352836	209584584	24.372115	8.406118
595	354025	210644875	24.392622	8.410833
596	355216	211708736	24.413111	8.415542
597	356409	212776173	24.433583	8.420246
598	357604	213847192	24.454039	8.424945
599	358801	214921799	24.474477	8.429638
600	360000	216000000	24.494897	8.434327
601	361201	217081801	24.515301	8.439010
602	362404	218167208	24.535688	8.443688
603	363609	219256227	24.556058	8.448360
604	364816	220348864	24.576412	8.453028
605	366025	221445125	24.596748	8.457691
606	367236	222545016	24.617067	8.462348
607	368449	223648543	24.637370	8.467000
608	369664	224755712	24.657656	8.471647
609	370881	225866529	24.677925	8.476289
610	372100	226981000	24.698178	8.480926

No.	Square.	Cube.	Square Root.	Cube Root.
611	373321	228099131	24.718414	8.485558
612	374544	229220928	24.738634	8.490185
613	375769	230346397	24.758837	8.494806
614	376996	231475544	24.779023	8.499423
615	378225	232608375	24.799194	8.504035
616	379456	233744896	24.819347	8.508642
617	380689	234885113	24.839485	8.513243
618	381924	236029032	24.859606	8.517840
619	383161	237176659	24.879711	8.522432
620	384400	238328000	24.899799	8.527019
621	385641	239483061	24.919872	8.531601
622	386884	240641848	24.939928	8.536178
623	388129	241804367	24.959968	8.540750
624	389376	242970624	24.979992	8.545317
625	390625	244140625	25.000000	8.549880
626	391876	245314376	25.019992	8.554437
627	393129	246491883	25.039968	8.558990
628	394384	247673152	25.059928	8.563538
629	395641	248858189	25.079872	8.568081
630	396900	250047000	25.099801	8.572619
631	398161	251239591	25.119713	8.577152
632	399424	252435968	25.139610	8.581681
633	400689	253636137	25.159491	8.586205
634	401956	254840104	25.179357	8.590724
635	403225	256047875	25.199206	8.595238
636	404496	257259456	25.219040	8.599748
637	405769	258474853	25.238859	8.604252
638	407044	259694072	25.258662	8.608753
639	408321	260917119	25.278449	8.613248
640	409600	262144000	25.298221	8.617739
641	410881	263374721	25.317978	8.622225
642	412164	264609288	25.337719	8.626706
643	413449	265847707	25.357445	8.631183
644	414736	267089984	25.377155	8.635655
645	416025	268336125	25.396850	8.640123
646	417316	269586136	25.416530	8.644585
647	418609	270840023	25.436195	8.649044
648	419904	272097792	25.455844	8.653497
649	421201	273359449	25.475478	8.657946
650	422500	274625000	25.495098	8.662391

No.	Square.	Cube.	Square Root.	Cube Root.
651	423801	275894451	25.514702	8.666831
652	425104	277167808	25.534291	8.671266
653	426409	278445077	25.553865	8.675697
654	427716	279726264	25.573424	8.680124
655	429025	281011375	25.592968	8.684546
656	430336	282300416	25.612497	8.688963
657	431649	283593393	25.632011	8.693376
658	432964	284890312	25.651511	8.697784
659	434281	286191179	25.670995	8.702188
660	435600	287496000	25.690465	8.706588
661	436921	288804781	25.709920	8.710983
662	438244	290117528	25.729361	8.715373
663	439569	291434247	25.748786	8.719760
664	440896	292754944	25.768198	8.724141
665	442225	294079625	25.787594	8.728519
666	443556	295408296	25.806976	8.732892
667	444889	296740963	25.826343	8.737260
668	446224	298077632	25.845696	8.741625
669	447561	299418309	25.865034	8.745985
670	448900	300763000	25.884358	8.750340
671	450241	302111711	25.903668	8.754691
672	451584	303464448	25.922963	8.759038
673	452929	304821217	25.942244	8.763381
674	454276	306182024	25.961510	8.767719
675	455625	307546875	25.980762	8.772053
676	456976	308915776	26.000000	8.776383
677	458329	310288733	26.019224	8.780708
678	459684	311665752	26.038433	8.785030
679	461041	313046839	26.057628	8.789347
680	462400	314432000	26.076810	8.793659
681	463761	315821241	26.095977	8.797968
682	465124	317214568	26.115130	8.802272
683	466489	318611987	26.134269	8.806572
684	467856	320013504	26.153394	8.810868
685	469225	321419125	26.172505	8.815160
686	470596	322828856	26.191602	8.819447
687	471969	324242703	26.210685	8.823731
688	473344	325660672	26.229754	8.828010
689	474721	327082769	26.248810	8.832285
690	476100	328509000	26.267851	8.836556

No.	Square.	Cube.	Square Root.	Cube Root.
691	477481	329939371	26.286879	8.840823
692	478864	331373888	26.305893	8.845085
693	480249	332812557	26.324893	8.849344
694	481636	334255384	26.343880	8.853598
695	483025	335702375	26.362853	8.857849
696	484416	337153536	26.381812	8.862095
697	485809	338608873	26.400758	8.866337
698	487204	340068392	26.419690	8.870576
699	488601	341532099	26.438608	8.874810
700	490000	343000000	26.457513	8.879040
701	491401	344472101	26.476405	8.883266
702	492804	345948408	26.495283	8.887488
703	494209	347428927	26.514147	8.891706
704	495616	348913664	26.532998	8.895920
705	497025	350402625	26.551836	8.900130
706	498436	351895816	26.570661	8.904337
707	499849	353393243	26.589472	8.908539
708	501264	354894912	26.608269	8.912737
709	502681	356400829	26.627054	8.916931
710	504100	357911000	26.645825	8.921121
711	505521	359425431	26.664583	8.925308
712	506944	360944128	26.683328	8.929490
713	508369	362467097	26.702060	8.933669
714	509796	363994344	26.720778	8.937843
715	511225	365525875	26.739484	8.942014
716	512656	367061696	26.758176	8.946181
717	514089	368601813	26.776856	8.950344
718	515524	370146232	26.795522	8.954503
719	516961	371694959	26.814175	8.958658
720	518400	373248000	26.832816	8.962809
721	519841	374805361	26.851443	8.966957
722	521284	376367048	26.870058	8.971101
723	522729	377933067	26.888659	8.975241
724	524176	379503424	26.907248	8.979377
725	525625	381078125	26.925824	8.983509
726	527076	382657176	26.944387	8.987637
727	528529	384240583	26.962938	8.991762
728	529984	385828352	26.981475	8.995883
729	531441	387420489	27.000000	9.000000
730	532900	389017000	27.018512	9.004113

No.	Square.	Cube.	Square Root.	Cube Root.
731	534361	390617891	27.037012	9.008223
732	535824	392223168	27.055499	9.012329
733	537289	393832837	27.073973	9.016431
734	538756	395446904	27.092434	9.020529
735	540225	397065375	27.110883	9.024624
736	541696	398688256	27.129320	9.028715
737	543169	400315553	27.147744	9.032802
738	544644	401947272	27.166155	9.036886
739	546121	403583419	27.184554	9.040965
740	547600	405224000	27.202941	9.045042
741	549081	406869021	27.221315	9.049114
742	550564	408518488	27.239677	9.053183
743	552049	410172407	27.258026	9.057248
744	553536	411830784	27.276363	9.061310
745	555025	413493625	27.294688	9.065368
746	556516	415160936	27.313001	9.069422
747	558009	416832723	27.331301	9.073473
748	559504	418508992	27.349589	9.077520
749	561001	420189749	27.367864	9.081563
750	562500	421875000	27.386128	9.085603
751	564001	423564751	27.404379	9.089639
752	565504	425259008	27.422618	9.093672
753	567009	426957777	27.440846	9.097701
754	568516	428661064	27.459060	9.101726
755	570025	430368875	27.477263	9.105748
756	571536	432081216	27.495454	9.109767
757	573049	433798093	27.513633	9.113782
758	574564	435519512	27.531800	9.117793
759	576081	437245479	27.549955	9.121801
760	577600	438976000	27.568098	9.125805
761	579121	440711081	27.586228	9.129806
762	580644	442450728	27.604348	9.133803
763	582169	444194947	27.622455	9.137797
764	583696	445943744	27.640550	9.141788
765	585225	447697125	27.658633	9.145774
766	586756	449455096	27.676705	9.149758
767	588289	451217663	27.694765	9.153737
768	589824	452984832	27.712813	9.157714
769	591361	454756609	27.730849	9.161687
770	592900	456533000	27.748874	9.165656

No.	Square.	Cube.	Square Root.	Cube Root.
771	594441	458314011	27.766887	9.169622
772	595984	460099648	27.784888	9.173585
773	597529	461889917	27.802878	9.177544
774	599076	463684824	27.820856	9.181500
775	600625	465484375	27.838822	9.185453
776	602176	467288576	27.856777	9.189402
777	603729	469097433	27.874720	9.193347
778	605284	470910952	27.892651	9.197290
779	606841	472729139	27.910572	9.201229
780	608400	474552000	27.928480	9.205164
781	609961	476379541	27.946377	9.209096
782	611524	478211768	27.964263	9.213025
783	613089	480048687	27.982137	9.216950
784	614656	481890304	28.000000	9.220873
785	616225	483736625	28.017852	9.224791
786	617796	485587656	28.035692	9.228707
787	619369	487443403	28.053520	9.232619
788	620944	489303872	28.071338	9.236528
789	622521	491169069	28.089144	9.240433
790	624100	493039000	28.106939	9.244335
791	625681	494913671	28.124722	9.248234
792	627264	496793088	28.142495	9.252130
793	628849	498677257	28.160256	9.256022
794	630436	500566184	28.178006	9.259911
795	632025	502459875	28.195744	9.263797
796	633616	504358336	28.213472	9.267680
797	635209	506261573	28.231188	9.271559
798	636804	508169592	28.248894	9.275435
799	638401	510082399	28.266588	9.279308
800	640000	512000000	28.284271	9.283178
801	641601	513922401	28.301943	9.287044
802	643204	515849608	28.319605	9.290907
803	644809	517781627	28.337255	9.294767
804	646416	519718464	28.354894	9.298624
805	648025	521660125	28.372522	9.302477
806	649636	523606616	28.390139	9.306328
807	651249	525557943	28.407745	9.310175
808	652864	527514112	28.425341	9.314019
809	654481	529475129	28.442925	9.317860
810	656100	531441000	28.460499	9.321697

No.	Square.	Cube.	Square Root.	Cube Root.
811	657721	533411731	28.478062	9.325532
812	659344	535387328	28.495614	9.329363
813	660969	537367797	28.513155	9.333192
814	662596	539353144	28.530685	9.337017
815	664225	541343375	28.548205	9.340839
816	665856	543338496	28.565714	9.344657
817	667489	545338513	28.583212	9.348473
818	669124	547343432	28.600699	9 352286
819	670761	549353259	28.618176	9.356095
820	672400	551368000	28.635642	9.359902
821	674041	553387661	28.653098	9.363705
822	675684	555412248	28.670542	9.367505
823	677329	557441767	28.687977	9.371302
824	678976	559476224	28.705400	9.375096
825	680625	561515625	28.722813	9.378887
826	682276	563559976	28.740216	9.382675
827	683929	565609283	28.757608	9.386460
828	685584	567663552	28.774989	9.390242
829	687241	569722789	28.792360	9.394021
830	688900	571787000	28.809721	9.397796
831	690561	573856191	28.827071	9.401569
832	692224	575930368	28.844410	9.405339
833	693889	578009537	28.861739	9.409105
834	695556	580093704	28.879058	9.412869
835	697225	582182875	28.896367	9.416630
836	698896	584277056	28.913665	9.420387
837	700569	586376253	28.930952	9.424142
838	702244	588480472	28.948230	9.427894
839	703921	590589719	28.965497	9.431642
840	705600	592704000	28.982754	9.435388
841	707281	594823321	29.000000	9.439131
842	708964	596947688	29.017236	9.442870
843	710649	599077107	29.034462	9.446607
844	712336	601211584	29.051678	9.450341
845	714025	603351125	29.068884	9.454072
846	715716	605495736	29.086079	9.457800
847	717409	607645423	29.103264	9.461525
848	719104	609800192	29 120440	9.465247
849	720801	611960049	29.137605	9.468966
850	722500	614125000	29.154760	9.472682

No.	Square.	Cube.	Square Root.	Cube Root.
851	724201	616295051	29.171904	9.476396
852	725904	618470208	29.189039	9.480166
853	727609	620650477	29.206164	9.483814
854	729316	622835864	29.223278	9.487518
855	731025	625026375	29.240383	9.491220
856	732736	627222016	29.257478	9.494919
857	734449	629422793	29.274562	9.498615
858	736164	631628712	29.291637	9.502308
859	737881	633839779	29.308702	9.505998
860	739600	636056000	29.325757	9.509685
861	741321	638277381	29.342802	9.513370
862	743044	640503928	29.359837	9.517051
863	744769	642735647	29.376862	9.520730
864	746496	644972544	29.393877	9.524406
865	748225	647214625	29.410882	9.528079
866	749956	649461896	29.427878	9.531750
867	751689	651714363	29.444864	9.535417
868	753424	653972032	29.461840	9.539082
869	755161	656234909	29.478806	9.542744
870	756900	658503000	29.495762	9.546403
871	758641	660776311	29.512709	9.550059
872	760384	663054848	29 529646	9.553712
873	762129	665338617	29.546573	9.557363
874	763876	667627624	29.563491	9.561011
875	765625	669921875	29.580399	9.564656
876	767376	672221376	29.597297	9.568298
877	769129	674526133	29 614186	9.571938
878	770884	676836152	29.631065	9.575574
879	772641	679151439	29.647934	9 579208
880	774400	681472000	29.664794	9.582840
881	776161	683797841	29.681644	9.586468
882	777924	686128968	29 698485	9.590094
883	779689	688465387	29.715316	9.593717
884	781456	690807104	29.732138	9.597337
885	783225	693154125	29.748950	9.600955
886	784996	695506456	29.765752	9.604570
887	786769	697864103	29.782545	9.608182
888	788544	700227072	29.799329	9.611791
889	790321	702595369	29.816103	9.615398
890	792100	704969000	29 832868	9.619002

No.	Square.	Cube.	Square Root.	Cube Root.
891	793881	707347971	29.849623	9.622603
892	795664	709732288	29.866369	9.626202
893	797449	712121957	29.883106	9.629797
894	799236	714516984	29.899833	9.633391
895	801025	716917375	29.916551	9.636981
896	802816	719323136	29.933259	9.640569
897	804609	721734273	29.949958	9.644154
898	806404	724150792	29.966648	9.647737
899	808201	726572699	29.983329	9.651317
900	810000	729000000	30.000000	9.654894
901	811801	731432701	30.016662	9.658468
902	813604	733870808	30.033315	9.662040
903	815409	736314327	30.049958	9.665610
904	817216	738763264	30.066593	9.669176
905	819025	741217625	30.083218	9.672740
906	820836	743677416	30.099834	9.676302
907	822649	746142643	30.116441	9.679860
908	824464	748613312	30.133038	9.683417
909	826281	751089429	30.149627	9.686970
910	828100	753571000	30.163206	9.690521
911	829921	756058031	30.182777	9.694069
912	831744	758550528	30.199338	9.697615
913	833569	761048497	30.215890	9.701158
914	835396	763551944	30.232433	9.704699
915	837225	766060875	30.248967	9.708237
916	839056	768575296	30.265492	9.711772
917	840889	771095213	30.282008	9.715305
918	842724	773620632	30.298515	9.718835
919	844561	776151559	30.315013	9.722636
920	846400	778688000	30.331502	9.725888
921	848241	781229961	30.347982	9.729411
922	850084	783777448	30.364453	9.732931
923	851929	786330467	30.380915	9.736448
924	853776	788889024	30.397368	9.739963
925	855625	791453125	30.413813	9.743476
926	857476	794022776	30.430248	9.746986
927	859329	796597983	30.446675	9.750493
928	861184	799178752	30.463092	9.753998
929	863041	801765089	30.479501	9.757500
930	864900	804357000	30.495901	9.761000

No.	Square.	Cube.	Square Root.	Cube Root.
931	866761	806954491	30.512293	9.764497
932	868624	809557568	30.528675	9.767992
933	870489	812166237	30.545049	9.771484
934	872356	814780504	30.561414	9.774974
935	874225	817400375	30.577770	9.778462
936	876096	820025856	30.594117	9.782947
937	877969	822656953	30.610456	9.785429
938	879844	825293672	30.626786	9.788909
939	881721	827936019	30.643107	9.792386
940	883600	830584000	30.659419	9.795861
941	885481	833237621	30.675723	9.799334
942	887364	835896888	30.692019	9.802804
943	889249	838561807	30.708305	9.806271
944	891136	841232384	30.724583	9.809736
945	893025	843908625	30.740852	9.813199
946	894916	846590536	30.757113	9.816659
947	896809	849278123	30.773365	9.820117
948	898704	851971392	30.789609	9.823572
949	900601	854670349	30.805844	9.827025
950	902500	857375000	30.822070	9.830476
951	904401	860085351	30.838288	9.833924
952	906304	862801408	30.854497	9.837369
953	908209	865523177	30.870698	9.840813
954	910116	868250664	30.886890	9.844254
955	912025	870983875	30.903074	9.847692
956	913936	873722816	30.919250	9.851128
957	915849	876467493	30.935417	9.854562
958	917764	879217912	30.951575	9.857993
959	919681	881974079	30.967725	9.861422
960	921600	884736000	30.983867	9.864848
961	923521	887503681	31.000000	9.868272
962	925444	890277128	31.016125	9.871694
963	927369	893056347	31.032241	9.875113
964	929296	895841344	31.048349	9.878530
965	931225	898632125	31.064449	9.881945
966	933156	901428696	31.080541	9.885357
967	935089	904231063	31.096624	9.888767
968	937024	907039232	31.112698	9.892175
969	938961	909853209	31.128765	9.895580
970	940900	912673000	31.144823	9.898983

No.	Square.	Cube.	Square Root.	Cube Root.
971	942841	915498611	31.160873	9.902383
972	944784	918330048	31.176915	9.905782
973	946729	921167317	31 192948	9.909178
974	948676	924010424	31.208973	9.912571
975	950625	926859375	31.224990	9.915962
976	952576	929714176	31.240999	9.919351
977	954529	932574833	31.256999	9.922738
978	956484	935441352	31.272992	9.926122
979	958441	938313739	31.288976	9.929504
980	960400	941192000	31.304952	9.932884
981	962361	944076141	31.320920	9.936261
982	964324	946966168	31.336879	9.939636
983	966289	949862087	31.352831	9.943009
984	968256	952763904	31.368774	9.946380
985	970225	955671625	31.384710	9.949748
986	972196	958585256	31.400637	9.953114
987	974169	961504803	31.416556	9.956477
988	976144	964430272	31.432467	9.959839
989	978121	967361669	31.448370	9.963198
990	980100	970299000	31.464265	9.966555
991	982081	973242271	31.480153	9.969909
992	984064	976191488	31.496032	9.973262
993	986049	979146657	31.511903	9.976612
994	988036	982107784	31.527766	9.979960
995	990025	985074875	31.543621	9.983305
996	992016	988047936	31.559468	9.986649
997	994009	991026973	31.575307	9.989990
998	996004	994011992	31.591138	9.993329
999	998001	997002999	31.606961	9.996666
1000	1000000	1000000000	31.62278	10.00000

2. Table of Ordinary Logarithms of Numbers to the Base 10.

From 1 to 100 with Characteristics.

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
1	0.000000	21	1.322219	41	1.612784	61	1.785330	81	1.908485
2	0.301030	22	1.342423	42	1.623249	62	1.792392	82	1.913814
3	0.477121	23	1.361728	43	1.633468	63	1.799341	83	1.919078
4	0.602060	24	1.380211	44	1.643453	64	1.806180	84	1.924279
5	0.698970	25	1.397940	45	1.653213	65	1.812913	85	1.929419
6	0.778151	26	1.414973	46	1.662758	66	1.819544	86	1.934498
7	0.845098	27	1.431364	47	1.672098	67	1.826075	87	1.939519
8	0.903090	28	1.447158	48	1.681241	68	1.832509	88	1.944483
9	0.954243	29	1.462398	49	1.690196	69	1.838849	89	1.949390
10	1.000000	30	1.477121	50	1.698970	70	1.845098	90	1.954243
11	1.041393	31	1.491362	51	1.707570	71	1.851258	91	1.959041
12	1.079181	32	1.505150	52	1.716003	72	1.857332	92	1.963788
13	1.113943	33	1.518514	53	1.724276	73	1.863323	93	1.968483
14	1.146128	34	1.531479	54	1.732394	74	1.869232	94	1.973128
15	1.176091	35	1.544068	55	1.740363	75	1.875061	95	1.977724
16	1.204120	36	1.556303	56	1.748188	76	1.880814	96	1.982271
17	1.230449	37	1.568202	57	1.755875	77	1.886491	97	1.986772
18	1.255273	38	1.579784	58	1.763428	78	1.892095	98	1.991226
19	1.278754	39	1.591065	59	1.770852	79	1.897627	99	1.995635
20	1.301030	40	1.602660	60	1.778151	80	1.903099	100	2.000000

TABLE OF ORDINARY LOGARITHMS TO THE BASE 10. FROM 100 TO 1000 WITHOUT CHARACTERISTICS

N.	0	1	2	3	4	5	6	7	8	9	D.
100	000000	000434	000868	001301	001734	002166	002598	003029	003461	003891	432
1	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
2	8600	9026	9451	9876	010300	010724	011147	011570	011993	012415	424
3	012837	013259	013680	014100	4521	4940	5360	5779	6197	6616	420
4	7033	7451	7868	8284	8700	9116	9532	9947	020361	020775	416
5	021189	021603	022016	022428	022841	023252	023664	024075	4486	4896	412
6	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
7	9384	9789	030195	030600	031004	031408	031812	032210	032619	033021	404
8	033424	033826	4227	4628	5029	5430	5830	6230	6629	7028	400
9	7426	7825	8223	8620	9017	9414	9811	040207	040602	040998	397
110	041393	041787	042182	042576	042969	043362	043755	044148	044540	044932	393
1	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	390
2	9218	9606	9993	050380	050766	051153	051538	051924	052309	052694	386
3	053078	053463	053846	4230	4613	4996	5378	5760	6142	6524	383
4	6905	7286	7666	8046	8426	8805	9185	9563	9942	060320	379
5	060698	061075	061452	061829	062206	062582	062958	063333	063709	4083	376
6	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	373
7	8186	8557	8928	9298	9668	070038	070407	070776	071145	071514	370
8	071882	072250	072617	072985	073352	3718	4085	4451	4816	5182	366
9	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	079181	079543	079904	080266	080626	080987	081347	081707	082067	082426	360
1	082785	083144	083503	3861	4219	4576	4934	5291	5647	6004	357
2	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
3	9905	090258	090611	090963	091315	091667	092018	092370	092721	093071	352
4	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
5	6910	7257	7604	7951	8298	8644	8990	9335	9681	100026	346
6	100371	100715	101059	101403	101747	102091	102434	102777	103119	3462	343
7	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	341
8	7210	7549	7888	8227	8565	8903	9241	9579	9918	110253	338
9	110590	110926	111263	111599	111934	112270	112605	112940	113275	3609	335
130	113943	114277	114611	114944	115278	115611	115943	116276	116608	116940	333
1	7271	7603	7934	8265	8595	8926	9256	9586	9915	120245	330
2	120574	120903	121231	121560	121888	122216	122544	122871	123198	3525	328
3	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
4	7105	7429	7753	8076	8399	8722	9045	9368	9690	130012	323
5	130334	130655	130977	131298	131619	131939	132260	132580	132900	3219	321
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8	170262	170555	170848	171141	171434	171726	172019	172311	172603	172895	293
9	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
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7	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
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7	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
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1	5307	5431	5555	5678	5802	5925	6049	6172	6296	6419	124
2	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652	123
3	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881	123
4	9003	9126	9249	9371	9494	9616	9739	9861	9984	550106	123
5	550228	550351	550473	550595	550717	550840	550962	551084	551206	1328	122
6	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547	122
7	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762	121
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4	561101	1221	1340	1459	1578	1698	1817	1936	2055	2174	119
5	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362	119
6	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	119
7	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
8	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909	118
9	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	118
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1	9374	9491	9608	9725	9842	9959	570076	570193	570309	570426	117
2	570543	570660	570776	570893	571010	571126	1243	1359	1476	1592	117
3	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
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5	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072	116
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5	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475	113
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7	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720	112
8	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838	112
9	9950	590061	590173	590284	590396	590507	590619	590730	590842	590953	112
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4	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487	110
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6	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681	110
7	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774	109
8	9883	9992	600101	600210	600319	600428	600537	600646	600755	600864	109
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2	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845	105
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5	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989	105
6	9093	9198	9302	9406	9511	9615	9719	9824	9928	620032	104
7	620136	620240	620344	620448	620552	620656	620760	620864	620968	1072	104
8	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	104
9	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146	104
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3	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	103
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5	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308	102
6	9410	9512	9613	9715	9817	9919	630021	630123	630224	630326	102
7	630428	630530	630631	930733	630835	630936	1038	1139	1241	1342	102
8	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356	101
9	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	101
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3	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390	100
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3	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	98
4	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	98
5	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	97
6	9335	9432	9530	9627	9724	9821	9919	650016	650113	650210	97
7	650308	650405	650502	650599	650696	650793	650890	0987	1084	1181	97
8	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150	97
9	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	97
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4	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	96
5	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870	95
6	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821	95
7	9916	660011	660106	660201	660296	660391	660486	660581	660676	660771	95
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2	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487	94
3	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424	94
4	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360	94
5	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293	93
6	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224	93
7	9317	9410	9503	9596	9689	9782	9875	9967	670060	670153	93
8	670246	670339	670431	670524	670617	670710	670802	670895	0988	1080	93
9	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005	93
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2	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769	92
3	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687	92
4	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	92
5	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	91
6	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427	91
7	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337	91
8	9428	9519	9610	9700	9791	9882	9973	680063	680154	680245	91
9	680336	680426	680517	680607	680698	680789	680879	0970	1060	1151	91
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4	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	90
5	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547	89
6	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	89
7	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	89
8	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	89
9	9309	9398	9486	9575	9664	9753	9841	9930	690019	690107	89
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2	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	88
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4	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	88
5	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394	88
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7	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	87
8	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	87
9	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883	87
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3	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	86
4	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205	86
5	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065	86
6	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	86
7	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	86
8	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	85
9	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	85
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8	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	84
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5	720159	720242	720325	720407	720490	720573	720655	720738	720821	0903	83
6	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	82
7	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	82
8	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	82
9	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	82
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4	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	81
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6	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	81
7	9974	730055	730136	730217	730298	730378	730459	730540	730621	730702	81
8	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508	81
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5	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113	80
6	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
7	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
8	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
9	9572	9651	9731	9810	9889	9968	740047	740126	740205	740284	79
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1	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	79
2	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	79
3	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
4	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
5	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	78
6	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
7	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
8	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	78
9	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	78
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3	750508	750586	750663	750740	0817	0894	0971	1048	1125	1202	77
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5	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
6	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	77
7	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	77
8	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
9	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	76
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1	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	76
2	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
3	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	76
4	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	76
5	9668	9743	9819	9894	9970	760045	760121	760196	760272	760347	75
6	760422	760498	760573	760649	760724	0799	0875	0950	1025	1101	75
7	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	75
8	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	75
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4	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	74
5	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	74
6	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
7	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	74
8	9377	9451	9525	9599	9673	9746	9820	9894	9968	770042	74
9	770115	770189	770263	770336	770410	770484	770557	770631	770705	0778	74
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1	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
2	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
3	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
4	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
5	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	73
6	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
7	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	73
8	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
9	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72
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3	780317	780389	780461	780533	780605	780677	780749	0821	0893	0965	72
4	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	72
5	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	72
6	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	72
7	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	71
8	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	71
9	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	71
610	785330	785401	785472	785543	785615	785686	785757	785828	785899	785970	71
1	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
2	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	71
3	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	71
4	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	71
5	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	71
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7	790285	790356	790426	790496	790567	790637	0707	0778	0848	0918	70
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9	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	70
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4	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	70
5	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	69
6	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
7	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
8	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	69
9	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69
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1	800029	800098	800167	800236	800305	800373	800442	800511	800580	800648	69
2	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	69
3	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
4	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	68
5	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
6	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
7	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
8	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
9	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	68
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2	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	68
3	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	67
4	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	67
5	9560	9627	9694	9762	9829	9896	9964	810031	810098	810165	67
6	810233	810300	810367	810434	810501	810569	810636	0703	0770	0837	67
7	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
8	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
9	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	67
650	812913	812980	813047	813114	813181	813247	813314	813381	813448	813514	67
1	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	67
2	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	67
3	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
4	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
5	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	66
6	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	66
7	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	66
8	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	66
9	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	66
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5	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	65
6	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	65
7	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	65
8	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	65
9	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	65
670	826075	826140	826204	826269	826334	826399	826464	826528	826593	826658	65
1	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	65
2	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	65
3	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	64
4	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	64
5	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	64
6	9947	830011	830075	830139	830204	830268	830332	830396	830460	830525	64
7	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166	64
8	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	64
9	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	64
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1	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	64
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4	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	63
5	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	63
6	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	63
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8	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	63
9	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
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5	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	62
6	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	62
7	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	62
8	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	62
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1	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	62
2	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	62
3	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	62
4	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	62
5	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	62
6	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
7	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	61
8	850033	850095	850156	850217	850279	850340	850401	850462	850524	850585	61
9	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	61
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1	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419	61
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3	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	61
4	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	61
5	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	61
6	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	61
7	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	61
8	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	60
9	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	60
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6	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	60
7	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	60
8	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	60
9	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	60
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1	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	59
2	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	59
3	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	59
4	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	59
5	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	59
6	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	59
7	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	59
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9	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	59
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1	9818	9877	9935	9994	870053	870111	870170	870228	870287	870345	59
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3	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	58
4	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	58
5	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	58
6	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	58
7	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	58
8	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	58
9	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	58
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1	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	58
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6	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	57
7	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	57
8	9669	9726	9784	9841	9898	9956	880013	880070	880127	880185	57
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1	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898	57
2	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	57
3	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	57
4	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	57
5	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172	57
6	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	57
7	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	57
8	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	57
9	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	56
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1	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	56
2	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	56
3	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	56
4	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	56
5	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	56
6	9862	9918	9974	890030	890086	890141	890197	890253	890309	890365	56
7	890421	890477	890533	0589	0645	0700	0756	0812	0868	0924	56
8	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	56
9	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	56
780	892095	892150	892206	892262	892317	892373	892429	892484	892540	892595	56
1	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	56
2	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	56
3	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	55
4	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	55
5	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	55
6	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
7	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	55
8	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	55
9	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
790	897627	897682	897737	897792	897847	897902	897957	898012	898067	898122	55
1	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	55
2	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	55
3	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	55
4	9821	9875	9930	9985	900039	900094	900149	900203	900258	900312	55
5	900367	900422	900476	900531	0586	0640	0695	0749	0804	0859	55
6	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	55
7	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	54
8	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
9	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	54
800	903090	903144	903199	903253	903307	903361	903416	903470	903524	903578	54
1	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	54
2	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	54
3	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	54
4	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
5	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	54
6	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
7	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	54
8	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
9	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
810	908485	908539	908592	908646	908699	908753	908807	908860	908914	908967	54
1	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	54
2	9556	9610	9663	9716	9770	9823	9877	9930	9984	910037	53
3	910091	910144	910197	910251	910304	910358	910411	910464	910518	0571	53
4	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	53
5	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	53
6	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
7	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
8	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
9	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761	53
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1	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	53
2	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	53
3	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	53
4	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	53
5	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	53
6	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	53
7	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	52
8	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	52
9	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
830	919078	919130	919183	919235	919287	919340	919392	919444	919496	919549	52
1	9601	9653	9706	9758	9810	9862	9914	9967	920019	920071	52
2	920123	920176	920228	920280	920332	920384	920436	920489	0541	0593	52
3	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	52
4	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
5	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	52
6	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	52
7	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	52
8	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	52
9	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	52
840	924279	924331	924383	924434	924484	924538	924589	924641	924693	924744	52
1	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	52
2	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	52
3	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
4	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	51
5	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	51
6	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
7	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	51
8	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	51
9	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	51
850	929419	929470	929521	929572	929623	929674	929725	929776	929827	929879	51
1	9930	9981	930032	930083	930134	930185	930236	930287	930338	930389	51
2	930440	930491	0542	0592	0643	0694	0745	0796	0847	0898	51
3	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	51
4	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	51
5	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	51
6	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	51
7	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	51
8	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	51
9	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	51
860	934498	934549	934599	934650	934700	934751	934801	934852	934902	934953	50
1	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	50
2	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	50
3	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	50
4	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	50
5	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468	50
6	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	50
7	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	50
8	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	50
9	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	50
870	939519	939569	939619	939669	939719	939769	939819	939869	939918	939968	50
1	940018	940068	940118	940168	940218	940267	940317	940367	940417	940467	50
2	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964	50
3	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	50
4	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	50
5	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455	50
6	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	50
7	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	49
8	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	49
9	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	49
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880	944483	944532	944581	944631	944680	944729	944779	944828	944877	944927	49
1	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	49
2	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	49
3	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	49
4	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	49
5	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385	49
6	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
7	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	49
8	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
9	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
890	949390	949439	949488	949536	949585	949634	949683	949731	949780	949829	49
1	9878	9926	9975	950024	950073	950121	950170	950219	950267	950316	49
2	950365	950414	950462	0511	0560	0608	0657	0706	0754	0803	49
3	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	49
4	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
5	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	48
6	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
7	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	48
8	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	48
9	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	48
900	954243	954291	954339	954387	954435	954484	954532	954580	954628	954677	48
1	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	48
2	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640	48
3	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	48
4	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
5	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080	48
6	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
7	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
8	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
9	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
910	959041	959089	959137	959185	959232	959280	959328	959375	959423	959471	48
1	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	48
2	9995	960042	960090	960138	960185	960233	960281	960328	960376	960423	48
3	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
4	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	48
5	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
6	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
7	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	47
8	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	47
9	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	47
920	963788	963835	963882	963929	963977	964024	964071	964118	964165	964212	47
1	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	47
2	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
3	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
4	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
5	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	47
6	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
7	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
8	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
9	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
930	968483	968530	968576	968623	968670	968716	968763	968810	968856	968903	47
1	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	47
2	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	47
3	9882	9928	9975	970021	970068	970114	970161	970207	970254	970300	47
4	970347	970393	970440	0486	0533	0579	0626	0672	0719	0765	46
5	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
6	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
7	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
8	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
9	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	46
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940	973128	973174	973220	973266	973313	973359	973405	973451	973497	973543	46
1	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	46
2	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
3	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
4	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
5	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
6	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
7	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
8	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
9	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
950	977724	977769	977815	977861	977906	977952	977998	978043	978089	978135	46
1	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
2	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
3	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
4	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
5	980003	980049	980094	980140	980185	980231	980276	980322	980367	980412	45
6	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
7	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
8	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
9	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
960	982271	982316	982362	982407	982452	982497	982543	982588	982633	982678	45
1	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
2	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
3	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
4	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
5	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
6	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
7	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
8	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
9	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
970	986772	986817	986861	986906	986951	986996	987040	987085	987130	987175	45
1	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
2	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
3	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
4	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
5	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	45
6	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
7	9895	9939	9983	990028	990072	990117	990161	990206	990250	990294	44
8	990339	990383	990428	0472	0516	0561	0605	0650	0694	0738	44
9	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
980	991226	991270	991315	991359	991403	991448	991492	991536	991580	991625	44
1	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
2	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
3	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
4	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
5	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
6	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
7	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
8	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
9	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	995635	995679	995723	995767	995811	995854	995898	995942	995986	996030	44
1	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
2	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
3	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
4	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
5	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
6	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
7	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
8	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
9	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43
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3. Table of Hyperbolic Logarithms of Numbers.

From 1 to 100.

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
1	0.000000	21	3.04452244	41	3.71357207	61	4.110873864	81	4.39444915
2	0.69314718	22	3.09104245	42	3.73766962	62	4.12713439	82	4.40671925
3	1.09861229	23	3.13549422	43	3.76120012	63	4.14313473	83	4.41884061
4	1.38629436	24	3.17805383	44	3.78418963	64	4.15888308	84	4.43081680
5	1.60943791	25	3.21887582	45	3.80666249	65	4.17438727	85	4.44265126
6	1.79175947	26	3.25809654	46	3.82864140	66	4.18965474	86	4.45434730
7	1.94591015	27	3.29583687	47	3.85014760	67	4.20469262	87	4.46590812
8	2.07944154	28	3.33220451	48	3.87120101	68	4.21950771	88	4.47733681
9	2.19722458	29	3.36729583	49	3.89182030	69	4.23410650	89	4.48863637
10	2.30258509	30	3.40119738	50	3.91202301	70	4.24849524	90	4.49980967
11	2.39789527	31	3.43398720	51	3.93182563	71	4.26267988	91	4.51085951
12	2.48490665	32	3.46573590	52	3.95124372	72	4.27666612	92	4.52178858
13	2.56494936	33	3.49650756	53	3.97029191	73	4.29045944	93	4.53259949
14	2.63905733	34	3.52636052	54	3.98898405	74	4.30406509	94	4.54329478
15	2.70805020	35	3.55534806	55	4.00733319	75	4.31748811	95	4.55387689
16	2.77258872	36	3.58351894	56	4.02535169	76	4.33073334	96	4.56434819
17	2.83321334	37	3.61091791	57	4.04305127	77	4.34380542	97	4.57471098
18	2.89037176	38	3.63758616	58	4.06044301	78	4.35670883	98	4.58496748
19	2.94443898	39	3.66356165	59	4.07753744	79	4.36944785	99	4.59511985
20	2.99573227	40	3.68887945	60	4.09434456	80	4.38202663	100	4.60517019

4. TABLES OF NATURAL SINES, COSINES, TANGENTS AND COTANGENTS FROM 0° TO 90°. READ THE TABLE DOWNWARDS FOR *SINES* AND UPWARDS FOR *COSINES*.

NATURAL SINES

′	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	′
0	000000	017452	034899	052336	069756	087156	104528	121869	139173	156434	60
5	1454	8907	6353	3788	071207	8605	5975	3313	140613	7871	55
10	2909	020361	7806	5241	2658	090053	7421	4756	2053	9307	50
15	4363	1815	9260	6693	4108	1502	8867	6199	3493	160743	45
20	5818	3269	040713	8145	5559	2950	110313	7642	4932	2178	40
25	7272	4723	2166	9597	7009	4398	1758	9084	6371	3613	35
30	8727	6177	3619	061049	8459	5846	3203	130526	7809	5048	30
35	010181	7631	5072	2500	9909	7293	4648	1968	9248	6482	25
40	1635	9085	6525	3952	071359	8741	6093	3411	150686	7916	20
45	3090	030539	7978	5403	2808	100188	7537	4851	2123	9350	15
50	4544	1992	9431	6854	4258	1635	8982	6292	3561	170783	10
55	5998	3446	050883	8306	5707	3082	120426	7733	4998	2216	5
60	7452	4899	2336	9756	7156	4528	1869	9173	6434	3648	0
Cos.	89°	88°	87°	86°	85°	84°	83°	82°	81°	80°	
Sin.	10°	11°	12°	13°	14°	15°	16°	17°	18°	19°	
0	173648	190809	207912	224951	241922	258819	275637	292372	309017	325568	60
5	5080	2237	9334	6368	3333	260224	7035	3762	310400	6943	55
10	6512	3664	210756	7784	4743	1628	8432	5152	1782	8317	50
15	7944	5090	2178	9200	6153	3031	9829	6542	3164	9691	45
20	9375	6517	3599	230616	7563	4434	281225	7930	4545	331063	40
25	180805	7943	5019	2031	8972	5837	2620	9318	5925	2435	35
30	2236	9368	6440	3445	250380	7238	4015	300706	7305	3807	30
35	3665	200793	7859	4859	1788	8640	5410	2093	8684	5178	25
40	5095	2218	9279	6273	3195	270040	6803	3479	320062	6547	20
45	6524	3642	220697	7686	4602	1440	8196	4864	1439	7917	15
50	7953	5065	2116	9098	6008	2840	9589	6249	2816	9285	10
55	9381	6489	3534	240510	7414	4239	290981	7633	4193	340653	5
60	190809	7912	4951	1922	8819	5637	2372	9017	5568	2020	0
Cos.	79°	78°	77°	76°	75°	74°	73°	72°	71°	70°	
Sin.	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°	
0	342020	358368	374607	390731	406737	422618	438371	453990	469472	484810	60
5	3387	9725	5955	2070	6065	3936	9678	5286	470755	6081	55
10	4752	361082	7302	3407	9392	5253	440984	6580	2038	7352	50
15	6117	2438	8649	4744	410719	6569	2289	7874	3320	8621	45
20	7481	3793	9994	6080	2045	7884	3593	9166	4600	9890	40
25	8845	5148	381339	7415	3369	9198	4896	460458	5880	491157	35
30	350207	6501	2683	8749	4693	430511	6198	1749	7159	2424	30
35	1569	7854	4027	400082	6016	1823	7499	3038	8436	3689	25
40	2931	9206	5369	1415	7338	3135	8799	4327	9713	4953	20
45	4291	370557	6711	2747	8660	4445	450098	5615	480989	6217	15
50	5651	1908	8052	4078	9980	5755	1397	6901	2263	7479	10
55	7010	3258	9392	5408	421300	7063	2694	8187	3537	8740	5
60	8368	4607	390731	6737	2618	8371	3990	9472	4810	500000	0
Cos.	69°	68°	67°	66°	65°	64°	63°	62°	61°	60°	
Sin.	30°	31°	32°	33°	34°	35°	36°	37°	38°	39°	
0	500000	515038	529919	544639	559193	573576	587785	601815	615661	629320	60
5	1259	6284	531152	5858	560398	4767	8061	2076	6807	630450	55
10	2517	7529	2384	7076	1602	5957	590336	4136	7951	1578	50
15	3774	8773	3615	8293	2805	7145	1310	5294	9094	2705	45
20	5030	520016	4844	9509	4007	8332	2482	6451	620235	3831	40
25	6285	1258	6072	550724	5207	9518	3653	7607	1376	4955	35
30	7538	2499	7300	1937	6406	580703	4823	8761	2515	6078	30
35	8791	3738	8526	3149	7604	1886	5991	9915	3652	7200	25
40	510043	4977	9751	4360	8801	3069	7159	611067	4789	8320	20
45	1293	6214	540974	5570	9997	4250	8325	2217	5923	9439	15
50	2543	7450	2197	6779	571191	5429	9489	3367	7057	640557	10
55	3791	8685	3419	7987	2384	6608	606653	4515	8189	1673	5
60	5038	9919	4639	9193	3576	7785	1815	5661	9320	2788	0
′	59°	58°	57°	56°	55°	54°	53°	52°	51°	50°	′

NATURAL COSINES

NATURAL SINES

′	40°	41°	42°	43°	44°	45°	46°	47°	48°	49°	′
0	642788	656059	669131	681998	694658	707107	719340	731354	743145	754710	60
5	3901	7156	670211	3061	5704	8134	720349	2345	4117	5663	55
10	5013	8252	1289	4123	6748	9161	1357	3334	5088	6615	50
15	6124	9346	2367	5183	7790	710185	2364	4323	6057	7565	45
20	7233	660439	3443	6242	8832	1209	3369	5309	7025	8514	40
25	8341	1530	4517	7299	9871	2230	4372	6294	7991	9461	35
30	9448	2620	5590	8355	700909	3250	5374	7277	8956	760406	30
35	650553	3709	6662	9409	1946	4269	6375	8259	9919	1350	25
40	1657	4796	7732	690462	2981	5286	7374	9239	750880	2292	20
45	2760	5882	8801	1513	4015	6302	8371	740218	1840	3232	15
50	3861	6966	9868	2563	5047	7316	9367	1195	2798	4171	10
55	4961	8049	680934	3611	6078	8329	730361	2171	3755	5109	5
60	6059	9131	1998	4658	7107	9340	1354	3145	4710	6044	0
Cos.	49°	48°	47°	46°	45°	44°	43°	42°	41°	40°	

Sin.	50°	51°	52°	53°	54°	55°	56°	57°	58°	59°	
0	766044	777146	788011	798636	809017	819152	829038	838671	848048	857167	60
5	6979	8060	8905	9510	9871	9985	9850	9462	8818	7915	55
10	7911	8973	9798	800383	810723	820817	830661	840251	9586	8662	50
15	8842	9884	790690	1254	1574	1647	1470	1039	850352	9406	45
20	9771	780794	1579	2123	2423	2475	2277	1825	1117	860149	40
25	770699	1702	2467	2991	3270	3302	3082	2609	1879	0890	35
30	1625	2608	3353	3857	4116	4126	3886	3391	2640	1629	30
35	2549	3513	4238	4721	4959	4949	4688	4172	3399	2366	25
40	3472	4416	5121	5584	5801	5770	5488	4951	4156	3102	20
45	4393	5317	6002	6445	6642	6590	6286	5728	4912	3836	15
50	5312	6217	6882	7304	7480	7407	7083	6503	5665	4567	10
55	6230	7114	7759	8161	8317	8223	7878	7277	6417	5297	5
60	7146	8011	8636	9017	9152	9038	8671	8048	7167	6025	0
Cos.	39°	38°	37°	36°	35°	34°	33°	32°	31°	30°	

Sin.	60°	61°	62°	63°	64°	65°	66°	67°	68°	69°	
0	866025	874620	882948	891007	898794	906308	913545	920505	927184	933580	60
5	6752	5324	3629	1665	9431	6922	4136	1072	7728	4101	55
10	7476	6026	4309	2323	900056	7533	4725	1638	8270	4619	50
15	8199	6727	4988	2979	0698	8143	5311	2201	8810	5135	45
20	8920	7425	5664	3633	1329	8751	5896	2762	9348	5650	40
25	9639	8122	6338	4284	1958	9357	6479	3322	9884	6162	35
30	870356	8817	7011	4934	2585	9961	7060	3880	930418	6672	30
35	1071	9510	7681	5582	3210	910563	7639	4435	0950	7181	25
40	1784	880201	8350	6229	3834	1164	8216	4989	1480	7687	20
45	2496	0891	9017	6873	4455	1762	8791	5541	2008	8191	15
50	3206	1578	9682	7515	5075	2358	9364	6090	2534	8694	10
55	3914	2264	890345	8156	5692	2953	9936	6638	3058	9194	5
60	4620	2948	1007	8794	6308	3545	920505	7184	3580	9693	0
Cos.	29°	28°	27°	26°	25°	24°	23°	22°	21°	20°	

Sin.	70°	71°	72°	73°	74°	75°	76°	77°	78°	79°	
0	939693	945519	951057	956305	961262	965926	970296	974370	978148	981627	60
5	940186	5991	1505	6729	1662	6301	0647	4696	8449	1904	55
10	0684	6462	1951	7151	2059	6675	0995	5020	8748	2178	50
15	1176	6930	2396	7571	2455	7046	1342	5342	9045	2450	45
20	1666	7397	2838	7990	2849	7415	1687	5662	9341	2721	40
25	2155	7861	3279	8406	3241	7782	2029	5980	9634	2989	35
30	2641	8324	3717	8820	3630	8148	2370	6296	9925	3255	30
35	3126	8784	4153	9232	4018	8511	2708	6610	980214	3519	25
40	3609	9243	4588	9642	4404	8872	3045	6921	0500	3781	20
45	4089	9699	5020	960050	4787	9231	3379	7231	0785	4041	15
50	4568	950154	5450	0456	5169	9588	3712	7539	1068	4298	10
55	5044	0606	5879	0860	5548	9943	4042	7844	1349	4554	5
60	5519	1057	6305	1262	5926	970296	4370	8148	1627	4808	0
′	19°	18°	17°	16°	15°	14°	13°	12°	11°	10°	′

NATURAL COSINES

′	80°	81°	82°	83°	84°	85°	86°	87°	88°	89°	′
0	984808	987688	990268	992546	994522	996195	997564	998630	999391	999848	60
5	5059	7915	0469	2722	4673	6320	7664	8705	9441	9872	55
10	5309	8139	0669	2896	4822	6444	7763	8778	9488	9894	50
15	5556	8362	0866	3068	4969	6566	7859	8848	9534	9914	45
20	5801	8582	1061	3238	5113	6685	7953	8917	9577	9932	40
25	6045	8800	1254	3406	5256	6802	8045	8984	9618	9948	35
30	6286	9016	1445	3572	5396	6917	8135	9048	9657	9962	30
35	6525	9230	1634	3735	5535	7030	8223	9111	9694	9974	25
40	6762	9442	1820	3897	5671	7141	8308	9171	9729	9983	20
45	6996	9651	2005	4056	5805	7250	8392	9229	9762	9990	15
50	7229	9859	2187	4214	5937	7357	8473	9285	9793	9996	10
55	7460	990065	2368	4369	6067	7462	8552	9339	9821	9999	5
60	7688	0268	2546	4522	6195	7564	8630	9391	9848	1.00000	0
Cos.	9°	8°	7°	6°	5°	4°	3°	2°	1°	0°	

NATURAL COSINES

NATURAL TANGENTS

′	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	′
0	000000	017455	034921	052408	069927	087489	105104	122785	140541	158384	60
5	1454	8910	6377	3866	071389	8954	6575	4261	2024	9876	55
10	2909	020365	7834	5325	2851	090421	8046	5738	3508	161368	50
15	4363	1820	9290	6784	4313	1887	9518	7216	4993	2860	45
20	5818	3275	040747	8243	5775	3354	110990	8694	6478	4354	40
25	7272	4731	2204	9703	7238	4822	2463	130173	7964	5848	35
30	8727	6186	3661	061163	8702	6289	3936	1652	9451	7343	30
35	010181	7641	5118	2623	080165	7757	5409	3132	150938	8838	25
40	1636	9097	6576	4083	1629	9226	6883	4613	2426	170334	20
45	3091	030553	8033	5543	3094	100695	8358	6094	3915	1831	15
50	4545	2009	9491	7004	4558	2164	9833	7576	5404	3329	10
55	6000	3465	050949	8465	6023	3634	121309	9058	6894	4828	5
60	7455	4921	2408	9927	7489	5104	2785	140541	8384	6327	0
Cot.	89°	88°	87°	86°	85°	84°	83°	82°	81°	80°	
Tan.	10°	11°	12°	13°	14°	15°	16°	17°	18°	19°	
0	176327	194380	212557	230868	249328	267949	286745	305731	324920	344328	60
5	7827	5890	4077	2401	250873	9509	8320	7322	6528	5955	55
10	9328	7401	5599	3934	2420	271069	9896	8914	8139	7585	50
15	180830	8912	7121	5469	3968	2631	291473	310508	9751	9216	45
20	2332	200425	8645	7004	5517	4194	3052	2104	331364	350848	40
25	3835	1938	220169	8541	7066	5759	4632	3701	2979	2483	35
30	5339	3452	1695	240079	8618	7325	6214	5299	4595	4119	30
35	6844	4967	3221	1618	260170	8892	7796	6899	6213	5756	25
40	8350	6483	4749	3157	1723	280460	9380	8500	7833	7396	20
45	9856	8000	6277	4698	3278	2029	300966	320103	9454	9037	15
50	191363	9518	7806	6241	4834	3600	2553	1707	341077	360680	10
55	2871	211037	9337	7784	6391	5172	4141	3313	2702	2324	5
60	4380	2557	230868	9328	7949	6745	5731	4920	4328	3970	0
Cot.	79°	78°	77°	76°	75°	74°	73°	72°	71°	70°	
Tan.	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°	
0	363970	383864	404026	424475	445229	466308	487733	509525	531709	554309	60
5	5618	5534	5719	6192	6973	8080	9534	511359	3577	6212	55
10	7268	7205	7414	7912	8719	9854	491339	3195	5447	8118	50
15	8920	8879	9111	9634	450467	471631	3145	5034	7319	560027	45
20	370573	390554	410810	431358	2218	3410	4955	6876	9195	1939	40
25	2228	2231	2511	3084	3971	5191	6767	8720	541074	3854	35
30	3885	3910	4214	4812	5726	6976	8582	520567	2956	5773	30
35	5543	5592	5919	6543	7484	8762	500399	2417	4840	7694	25
40	7204	7275	7626	8276	9244	480551	2219	4270	6728	9619	20
45	8866	8960	9335	440011	461006	2343	4042	6126	8619	571547	15
50	380530	400646	421046	1748	2771	4137	5867	7984	550513	3478	10
55	2196	2335	2759	3487	4538	5933	7695	9845	2409	5413	5
60	3864	4026	4475	5229	6308	7733	9525	531709	4309	7350	0
′	69°	68°	67°	66°	65°	64°	63°	62°	61°	60°	′

NATURAL COTANGENTS

NATURAL TANGENTS

′	30°	31°	32°	33°	34°	35°	36°	37°	′
0	577350	600861	624869	649408	674509	700208	726543	753554	60
5	9091	2842	6894	651477	6627	2377	6767	5837	55
10	581235	4827	8921	3551	8749	4551	730996	8125	50
15	3183	6815	630953	5629	680876	6730	3230	760418	45
20	5134	8807	2988	7710	3007	8913	5469	2716	40
25	7088	610802	5027	9796	5142	711101	7713	5019	35
30	9045	2801	7070	661886	7281	3293	9961	7327	30
35	591006	4803	9117	3979	9425	5430	742214	9640	25
40	2970	6809	641167	6077	691572	7691	4472	771959	20
45	4938	8819	3222	8179	3725	9897	6735	4283	15
50	6908	620832	5280	670285	5881	722108	9003	6612	10
55	8883	2849	7342	2394	8042	4323	751276	8946	5
60	600861	4869	9408	4509	700208	6543	3554	781286	0
Cot.	59°	58°	57°	56°	55°	54°	53°	52°	
Tan.	38°	39°	40°	41°	42°	43°	44°	45°	
0	781286	809784	839100	869287	900404	932515	965689	1.000000	60
5	3631	812195	841581	871844	3041	5238	8504	002013	55
10	5981	4612	4069	4407	5685	7968	971326	005835	50
15	8336	7034	6563	6977	8336	940706	4157	008765	45
20	790698	9463	9062	9553	910994	3451	6996	011704	40
25	3064	821897	851568	882136	3659	6204	9842	014651	35
30	5436	4336	4081	4725	6331	8965	982697	017607	30
35	7813	6782	6599	7322	9010	951733	5560	020572	25
40	800196	9234	9124	9924	921697	4508	8432	023546	20
45	2585	831691	861655	892534	4391	7292	991311	026529	15
50	4979	4155	4193	5151	7091	960083	4199	029520	10
55	7379	6624	6736	7774	9800	2882	7095	032521	5
60	9784	9100	9287	900404	932515	5689	1.000000	035530	0
Cot.	51°	50°	49°	48°	47°	46°	45°	44°	
Tan.	46°	47°	48°	49°	50°	51°	52°	53°	
0	1.035530	1.072369	1.110613	1.150368	1.191754	1.234897	1.279942	1.327045	60
5	038549	075501	113866	153753	195280	238576	283786	331068	55
10	041577	078642	117131	157150	198818	242269	287645	335098	50
15	044614	081794	120405	160557	202360	245974	291512	339162	45
20	047660	084955	123691	163976	205933	249693	295406	343233	40
25	050715	088127	126987	167407	209509	253426	299308	347320	35
30	053780	091309	130294	170850	213097	257172	303225	351422	30
35	056854	094500	133612	174304	216698	260932	307158	355541	25
40	059938	097702	136941	177770	220312	264706	311105	359676	20
45	063031	100914	140282	181248	223939	268494	315067	363828	15
50	066134	104137	143633	184738	227579	272296	319044	367996	10
55	069247	107369	146995	188240	231231	276112	323037	372181	5
60	072369	110613	150368	191754	234897	279942	327045	376382	0
Cot.	44°	43°	41°	40°	39°	38°	37°	36°	
Tan.	54°	55°	56°	57°	58°	59°	60°	61°	
0	1.376382	1.428148	1.482561	1.539865	1.600335	1.664280	1.732051	1.804048	60
5	380600	432578	487222	544779	605526	669776	737883	810252	55
10	384835	437027	491904	549716	610742	675299	743745	816489	50
15	389088	441494	496606	554674	615982	680849	749637	822759	45
20	393357	445980	501328	559655	621247	686426	755559	829063	40
25	397644	450485	506071	564659	626537	692031	761511	835400	35
30	401948	455009	510835	569686	631852	697663	767494	841771	30
35	406270	459552	515620	574735	637192	703323	773508	848176	25
40	410610	464115	520426	579808	642558	709012	779552	854616	20
45	414967	468697	525254	584904	647949	714728	785629	861091	15
50	419343	473298	530102	590024	653366	720474	791736	867600	10
55	423736	477920	534973	595167	658810	726248	797876	874146	5
60	428148	482561	539865	600335	664280	732051	804048	880727	0
′	35°	34°	33°	32°	31°	30°	29°	28°	′

NATURAL COTANGENTS

NATURAL TANGENTS

′	62°	63°	64°	65°	66°	67°	68°	69°	′
0	1.880727	1.962611	2.050304	2.144507	2.246037	2.355852	2.475087	2.605089	60
5	887344	969687	057895	152676	254857	365412	485489	616457	55
10	893997	976805	065532	160896	263716	375037	495666	627012	50
15	900687	983964	073215	169168	272673	384729	505920	639455	45
20	907415	991164	080944	177492	281669	394489	517151	651087	40
25	914179	998406	088720	185869	290726	404317	527860	662809	35
30	920982	2.005690	096544	194300	299843	414214	538648	674622	30
35	927823	013016	104415	202784	309021	424180	549516	686527	25
40	934702	020386	112335	211323	318261	434217	560465	698525	20
45	941620	027799	120303	219918	327563	444326	571496	710619	15
50	948577	035257	128321	228568	336929	454506	582609	722808	10
55	955574	042758	136389	237274	346358	464760	593807	735093	5
60	962611	050304	144507	246037	355852	475087	605089	747477	0
Cot.	27°	26°	25°	24°	23°	22°	21°	20°	
Tan.	70°	71°	72°	73°	74°	75°	76°	77°	
0	2.747477	2.904211	3.077684	3.270853	3.487414	3.732051	4.010781	4.331476	60
5	759961	917991	092083	287949	506656	753882	035778	360400	55
10	772545	931889	108421	305209	526094	775952	061070	389694	50
15	785231	945905	123999	322658	545733	798266	086663	419364	45
20	798020	960042	139719	340233	565575	820828	112561	449418	40
25	810913	974302	155584	358001	585624	843642	138772	479864	35
30	823913	988685	171595	375943	605884	866713	165300	510709	30
35	837020	3.003194	187754	394063	626357	890045	192151	541961	25
40	850235	017830	204064	412363	647047	913642	219332	573629	20
45	863560	032595	220526	430845	667958	937509	246848	605721	15
50	876997	047492	237144	449512	689093	961652	274707	638246	10
55	890547	062520	253918	468368	710456	986074	302914	671212	5
60	904211	077684	270853	487414	732051	4.010781	331476	704630	0
Cot.	19°	18°	17°	16°	15°	14°	13°	12°	
Tan.	78°	79°	80°	81°	82°	83°	84°	85°	
0	4.704630	5.144554	5.671282	6.313752	7.115370	8.144346	9.514365	11.43005	60
5	738508	184804	719917	373736	191246	243449	649348	62476	55
10	772857	225665	769369	434843	268726	344956	788173	82617	50
15	807685	267152	819657	497104	347861	448957	931009	12.03462	45
20	843005	309279	870804	560554	428766	555547	10.07803	25051	40
25	878825	352053	922832	625296	511318	664822	22043	47422	35
30	915157	395517	975764	691356	595754	776887	38540	70620	30
35	952013	439659	6.029625	758783	682077	891851	54615	94692	25
40	989403	484505	084438	826644	770351	9.009826	71191	13.19688	20
45	5.027340	530072	140230	896880	860642	130935	88292	45663	15
50	065835	576379	197028	968234	953022	255304	11.05943	72674	10
55	104902	623442	254859	7.041048	8.047565	383066	24171	14.00786	5
60	144554	671282	313752	115370	144346	514365	43005	30067	0
Cot.	11°	10°	9°	8°	7°	6°	5°	4°	
Tan.	86°	Diff.	87°	Diff.	88°	Diff.	89°	Diff.	
0	14.30067		19.08114		28.63625		57.28996		60
5	60592	30525	62730	54616	29.88230	1.24605	62.49915	5.20919	55
10	92442	31850	20.20555	57825	31.24158	1.35928	68.75009	6.25094	50
15	15.25705	33263	81883	61328	32.73026	1.48868	76.39001	7.63902	45
20	60478	34773	21.47040	65157	34.36777	1.63751	85.93979	9.54978	40
25	96867	36389	22.16398	69358	36.17760	1.80983	98.21794	12.2782	35
30	16.34986	38119	90317	73979	38.18846	2.01086	114.5887	16.3708	30
35	74961	39975	23.69454	79077	40.43584	2.24738	137.5075	22.9188	25
40	17.16934	41973	24.54176	84722	42.96408	2.52824	171.8854	34.3779	20
45	61056	44122	25.45170	90994	45.82935	2.86527	229.1817	57.2963	15
50	18.07498	46442	26.43160	97990	49.10388	3.27453	343.7737	114.5920	10
55	56447	48949	27.48985	1.05825	52.88211	3.77823	687.5489	343.7752	5
60	19.08114	51667	28.63625	1.14640	57.28996	4.40785	Infinite.	Infinite.	0
′	3°	Diff.	2°	Diff.	1°	Diff.	0°	Diff.	′

NATURAL COTANGENTS

5. Table of Circular and Hyperbolic Functions of Radians.

Angle u in Circular Measure.	Sine u.	Cosine u.	Sinh u.	Cosh u.
0	0	1	0	1
0.1	0.09983	0.99500	0.10017	1.00500
0.2	0.19867	0.98007	0.20134	1.02007
0.3	0.29552	0.95534	0.30452	1.04534
0.4	0.38942	0.92106	0.41075	1.08107
0.5	0.47943	0.87758	0.52110	1.12763
0.6	0.56464	0.82534	0.63665	1.18547
0.7	0.64422	0.76484	0.75858	1.25517
0.8	0.71736	0.69671	0.88811	1.33743
0.9	0.78333	0.62161	1.02652	1.43309
1.0	0.84147	0.54030	1.17520	1.54308
1.1	0.89120	0.45360	1.33565	1.66852
1.2	0.93204	0.36236	1.50946	1.81066
1.3	0.96356	0.26750	1.69838	1.97091
1.4	0.98545	0.16997	1.90430	2.15090
1.5	0.99749	0.07074	2.12928	2.35241
1.57	1.00000	0.0000	2.29930	2.50735
1.6	0.99957	−0.02920	2.37557	2.57746
1.7	0.99167	−0.12880	2.64563	2.82832
1.8	0.97383	−0.22728	2.94217	3.10747
1.9	0.94627	−0.32343	3.26816	3.41773
2.0	0.90926	−0.41619	3.62686	3.76220
2.1	0.86316	−0.50493	4.02186	4.14431
2.2	0.80845	−0.58858	4.45711	4.56791
2.3	0.74564	−0.66635	4.93696	5.03722
2.4	0.67539	−0.73743	5.46623	5.55095
2.5	0.59840	−0.80120	6.05020	6.13229
2.6	0.51542	−0.85694	6.69473	6.76900
2.7	0.42730	−0.90411	7.40626	7.47347
2.8	0.33396	−0.94225	8.19192	8.25273
2.9	0.23818	−0.97098	9.05956	9.11458
3.0	0.14100	−0.99000	10.0179	10.0677
3.1	0.04159	−0.99913	11.0765	11.1215
3.1416	0.0000	−1.0000	—	—
3.2	−0.05837	−0.99830	12.2459	12.2866
3.3	−0.15734	−0.98746	13.5379	13.5747
3.4	−0.25555	−0.96677	14.9654	14.9987
3.5	−0.35090	−0.93643	16.5426	16.5728
3.6	−0.44261	−0.89673	18.2855	18.3128
3.7	−0.52991	−0.84806	20.2213	20.2360
3.8	−0.61194	−0.79090	22.3394	22.3618
3.9	−0.68710	−0.72657	24.6911	24.7113

Note.—A unit angle in Circular Measure is called 1 *Radian* and is equal to $360/2\pi = 57^\circ\ 17'\ 45''$ nearly.

5. Table of Circular and Hyperbolic Functions of Radians (*continued*).

Angle u in Circular Measure.	Sine u.	Cosine u.	Sinh u.	Cosh u.
4.0	− 0.75687	− 0.65358	27.2899	27.3082
4.1	− 0.81834	− 0.57474	30.1619	30.1784
4.2	− 0 87163	− 0.49106	33.3357	33.3507
4.3	− 0.91620	− 0.40071	36.8431	36.8567
4.4	− 0.95160	− 0.30727	40.7190	40.7316
4.5	− 0.97752	− 0.21080	45.0030	45.0141
4.6	− 0.99370	− 0.11216	49.7371	49.7472
4.7	− 0.99992	− 0.01240	54.9690	54.9781
4.712	− 1.0000	0.00000	—	—
4.8	− 0.99616	0.08748	60.7511	60.7593
4.9	− 0.98245	0.18649	67.1412	67.1486
5.0	− 0.95895	0.28364	74.2032	74.2099
5.1	− 0.92582	0.37797	82.0079	82.0140
5.2	− 0.88346	0.46850	90.6333	90.6388
5.3	− 0.83228	0.55436	100.167	100.171
5.4	− 0.77275	0.63473	110.701	110.705
5.5	− 0.70527	0.70864	122.344	122.348
5.6	− 0.63128	0.77556	135.211	135.215
5.7	− 0.55070	0.83470	149.432	149.435
5.8	− 0.46462	0.88553	165.148	165.151
5.9	− 0.37389	0.92748	182.517	182.520
6.0	− 0.27944	0.96017	201.713	201.716
6.1	− 0.18218	0.98325	223	223
6.2	− 0.08310	0.99654	246.5	246.5
6.283	0.0000	1.0000	—	—
6.3	0.01679	0.99986	272.5	272.5
6.4	0.11652	0.99318	301	301
6.5	0.21510	0.97658	332.5	332.5
6.6	0.31151	0.95024	367.5	367.5
6.7	0.40480	0.91439	406	406
6.8	0.49410	0.86941	449	449
6.9	0.57842	0.81575	497.5	497.5
7.0	0.65700	0.75390	550	550
7.1	0.72897	0.68455	606	606
7.2	0.79366	0.60836	670	670
7.3	0.85042	0.52608	740	740
7.4	0.89870	0.43856	817	817

Note.—A unit angle in Circular Measure is called 1 *Radian* and is equal to $360/2\pi = 57^\circ\ 17'\ 45''$ nearly.

5. TABLE OF CIRCULAR AND HYPERBOLIC FUNCTIONS OF RADIANS (*continued*).

Angle u in Circular Measure.	Sine u.	Cosine u.	Sinh u.	Cosh u.
7·5	0.93799	0.34667	905	905
7·6	0.96793	0.25130	997·5	997·5
7·7	0.98811	0.15340	1105	1105
7·8	0.99854	0.05400	1220	1220
7·9	0.99894	− 0.04593	1348	1348
8·0	0.98936	− 0.14548	1490	1490
8·1	0.96990	− 0.24353	1646	1646
8·2	0.94075	− 0.33915	1820	1820
8·3	0.90220	− 0.43136	2010	2010
8·4	0.85460	− 0.51927	2223	2223
8·5	0.79852	− 0.60200	2457	2457
8·6	0.73442	− 0.67870	2711	2711
8·7	0.66295	− 0.74860	3001	3001
8·8	0.58495	− 0.81108	3316	3316
8·9	0.50102	− 0.86542	3666	3666
9·0	0.41212	− 0.91112	4057	4057
9·1	0.31911	− 0.94770	4475	4475
9·2	0.22300	− 0.97482	4947	4947
9·3	0.12447	− 0.99222	5467	5467
9·4	0.02480	− 0.99969	6043	6043

Note.—A unit angle in Circular Measure is called 1 *Radian* and is equal to $360/2\pi = 57^\circ\ 17'\ 45''$ nearly.

6. Table of Wire Gauges in Common Use.

S.W.G. = Standard Wire Gauge.
B.W.G. = Birmingham Wire Gauge.
B. & S. = Brown and Sharpe Gauge or American Wire Gauge.

No.	S.W.G.	B.W.G.	B. & S.	No.	S.W.G.	B.W.G.	B. & S.	No.	S.W.G.	B.W.G.	B. & S.
	Inch.	Inch.	Inch.		Inch.	Inch.	Inch.		Inch.	Inch.	Inch.
4/0	.400	.454	.4600	15	.072	.072	.0571	33	.0100	.008	.0071
3/0	.372	.425	.4096	16	.064	.065	.0508	34	.0092	.007	.0063
2/0	.348	.380	.3648	17	.056	.058	.0453	35	.0084	.005	.0056
0	.324	.340	.3249	18	.048	.049	.0403	36	.0076	.024	.0050
1	.300	.300	.2893	19	.040	.042	.0350	37	.0068		.0045
2	.276	.284	.2576	20	.036	.035	.0320	38	.0060		.0040
3	.252	.259	.2294	21	.032	.032	.0285	39	.0052		.0035
4	.232	.238	.2043	22	.028	.028	.0253	40	.0048		.0031
5	.212	.220	.1819	23	.024	.025	.0226	41	.0044		
6	.192	.203	.1620	24	.022	.022	.0201	42	.0040		
7	.176	.180	.1443	25	.020	.020	.0176	43	.0036		
8	.160	.165	.1285	26	.018	.018	.0159	44	.0032		
9	.144	.148	.1144	27	.0164	.016	.0142	45	.0028		
10	.128	.134	.1019	28	.0148	.014	.0126	46	.0024		
11	.116	.120	.0907	29	.0136	.013	.0113	47	.0020		
12	.104	.109	.0808	30	.0124	.012	.0100	48	.0016		
13	.092	.095	.0720	31	.0116	.010	.0089	49	.0012		
14	.080	.083	.0641	32	.0108	.009	.0079	50	.0010		

Table of Equivalents.

Pounds per Kilometre = Pounds per 1000 Yards plus 10 per cent.
Kilogrammes per Kilometre = Pounds per 1000 Yards divided by 2.

1 Inch	= 25.39 m/m.	1 Millimetre	= .0394 Inch.
1 Square Inch	= 645.2 Square m/m.	1 Sq. Millimetre	= .00155 Square Inch.
1 Cubic Inch	= 16.387 Cubic c/m.	1 Cubic ,,	= .061027 Cubic Inch.
1 Pound (avoir.)	= .454 Kilogramme.	1 Kilogramme	= 2.204 lbs. (avoir.).
1 Mile	= 1.609 Kilometre.	1 Kilometre	= 1094 Yards.

Useful Formulæ and Data.

Copper.—Weighs .321 lbs. per cubic inch, or 555 lbs. per cubic foot, and has a specific gravity 8.912 at 60° F.
Tensile strength of hard drawn wire is 25 to 30 tons per square inch section.
Resistance per mil foot 10.2044 ohms.
Weight per mile in lbs. = D^2 (in mils) ÷ 62.55, or area in square inches × 20,350.
Temperature co-efficient = .00238 per degree F. (Messrs. Clarke, Forde, & Taylor).

Eureka.—Resistance per mil foot 285.723 ohms.

Note.—By kind permission of the London Electric Wire Company, Ltd., the above Tables are taken from their Trade Publications.

7. TABLE OF RESISTANCES AND WEIGHTS OF VARIOUS SIZES OF HIGH CONDUCTIVITY COPPER WIRE.
By kind permission of The London Electric Wire Company, Ltd., Playhouse Yard, Golden Lane, London, E.C.

SIZE.	DIAMETER.		SECTIONAL AREA.		WEIGHT.		RESISTANCE AT 60° F.		CURRENT CAPACITY.	
					POUNDS.		Standard Ohms.		Amperes @	
S.W.G.	Inch.	M/m.	Square Inch.	Square M/m.	Per 1000 Yards.	Per Mile.	Per 1000 Yards.	Per Mile.	1000 per Sq. Inch.	I.E.E.
47	.0020	.0508	.0000031	.002027	.0363	.0640	7653	13469	.0031	
46	.0024	.0610	.0000045	.002919	.0523	.0921	5313	9351	.0045	
45	.0028	.0711	.0000062	.003973	.0712	.1253	3905	6873	.0062	
44	.0032	.0813	.0000080	.005188	.0930	.1637	2988	5262	.0080	
43	.0036	.0914	.0000102	.006567	.1177	.2072	2362	4157	.0102	
42	.0040	.1016	.0000126	.008109	.1453	.2558	1913	3367	.0126	
41	.0044	.1118	.0000152	.009810	.1758	.3093	1581	2783	.0152	
40	.0048	.1219	.0000181	.011674	.2093	.3683	1328	2338	.0181	
39	.0052	.1321	.0000212	.013701	.2451	.4314	1132	1992	.0212	
38	.0060	.1524	.0000283	.018241	.3272	.5759	850.3	1497	.0283	
37	.0068	.1727	.0000363	.023430	.4197	.7387	662.0	1165	.0363	
36	.0076	.1930	.0000454	.029267	.5245	.9232	530.0	932.8	.0454	
35	.0084	.2134	.0000554	.035752	.6406	1.127	433.9	763.6	.0554	
34	.0092	.2337	.0000665	.042887	.7688	1.353	361.7	636.6	.0665	
33	.0100	.2540	.0000785	.050670	.9085	1.598	306.1	538.8	.0785	
32	.0108	.2743	.0000916	.059102	1.059	1.863	262.5	461.9	.0916	
31	.0116	.2946	.000106	.068181	1.222	2.151	227.5	400.4	.1057	
30	.0124	.3149	.000121	.077910	1.399	2.462	199.1	350.4	.1207	

29	.0136	.3454	.000145	.093722	1.676	2.950	165.5	291.3	.1453	
28	.0148	.3759	.000172	.11099	1.989	3.500	139 8	246.0	.1720	
27	.0164	.4166	.000211	.13628	2.440	4.294	113.8	200.3	.2112	
26	.018	.4572	.000254	.1642	2.942	5.18	94 48	166.3	.254	
25	.020	.5080	.000314	.2027	3.633	6.39	76.53	134.7	.314	
24	.022	.5588	.000380	.2453	4.392	7.73	63.24	111.3	.380	
23	.024	.6096	.000452	.2919	5.233	9.21	53.13	93.51	.452	
22	.028	.7112	.000616	.3973	7.120	12.53	39.05	68.72	.616	1.7
21	.032	.8128	.000804	.5188	9.301	16.37	29.90	52.62	.804	2.2
20	.036	.9144	.001018	.6567	11.77	20.72	23.62	41.57	1.018	2.6
19	.040	1.016	.001257	.8109	14.53	25.58	19.13	33.67	1.257	3.2
18	.048	1.219	.001810	1.168	20.93	36.83	13.28	23.38	1.810	4.2
17	.056	1.422	.002463	1.589	28.48	50.12	9.762	17.18	2.463	5.4
16	.064	1.626	.003217	2.075	37.20	65.47	7.478	13.16	3.217	6.8
15	.072	1.829	.004072	2.627	47.09	82.87	5.904	10.39	4.072	8.2
14	.080	2.032	.005027	3.243	58.13	102.3	4.784	8.419	5.027	9.8
13	.092	2.337	.006648	4.289	76.88	135.3	3.617	6.366	6.648	12.4
12	.104	2.642	.008495	5.480	98.24	172.9	2.831	4.982	8.495	15.0
11	.116	2.946	.01057	6.819	122.2	215.1	2.275	4.004	10.57	18.0
10	.128	3.251	.01287	8.303	148.8	261.9	1.868	3.288	12.87	21.0
9	.144	3.658	.01629	10.51	188.4	331.5	1.476	2.598	16.29	27.0
8	.160	4.064	.02011	12.97	232.5	409.2	1.195	2.104	20.11	31.0
7	.176	4.470	.02433	15.70	281.3	495.1	.9881	1.739	24 33	36 0
6	.192	4.877	.02895	18.68	334.7	589.1	.8307	1.462	28.95	42.0

Note.—This Table is compiled according to the Standards fixed by the Institution Committee on Copper Conductors.

8. Table of Resistances and Weights of Various Sizes of Aluminium Wire.

By kind permission of the British Aluminium Company, Ltd., 109 Queen Victoria Street, London, E.C.

Size of Wire. S.W.G.	Current 600 amp. per sq. in.	Diameter of Wire.		Sectional Area of Wire.		Resistance in Ohms per 1000 yds.		Resistance in Ohms per Statute Mile.		Weight.	
		In.	Mm.	Sq. in.	Sq. mm.	Soft.	Hard.	Soft.	Hard.	Lbs. 1000 yds.	Lbs. mile.
7/0	118	.500	12.7	.196	127.	0.204	0.208	0.359	0.366	692	1220
6/0	101	.464	11.8	.163	109	0.237	0.242	0.417	0.425	596	1050
5/0	88.0	.432	11.0	.147	94.6	0.273	0.279	0.481	0.491	517	909
4/0	75.4	.400	10.2	.126	81.1	0.319	0.325	0.561	0.572	443	780
3/0	65.2	.372	9.45	.109	70.1	0.369	0.376	0.649	0.662	382	674
2/0	57.1	.348	8.84	.0951	61.4	0.421	0.430	0.742	0.756	335	590
1/0	49.5	.324	8.23	.0825	53.2	0.486	0.496	0.855	0.872	291	511
1	42.4	.300	7.62	.0707	45.6	0.567	0.577	0.998	1.02	249	438
2	35.9	.276	7.01	.0598	38.6	0.670	0.683	1.18	1.20	211	371
3	29.9	.252	6.40	.0499	32.2	0.803	0.820	1.41	1.44	176	309
4	25.4	.232	5.89	.0423	27.3	0.948	0.967	1.67	1.70	149	262
5	21.2	.212	5.38	.0353	22.8	1.13	1.16	2.00	2.04	124	219
6	17.4	.192	4.88	.0289	18.7	1.38	1.41	2.44	2.48	102	179
7	14.6	.176	4.47	.0243	15.7	1.65	1.68	2.90	2.96	85.7	151
8	12.1	.160	4.06	.0201	13.0	1.99	2.03	3.51	3.58	70.9	125
9	9.77	.144	3.66	.0163	10.5	2.46	2.51	4.33	4.42	57.4	101
10	7.72	.128	3.25	.0129	8.30	3.11	3.18	5.48	5.59	45.4	79.8
11	6.34	.116	2.95	.0106	6.82	3.79	3.87	6.67	6.81	37.3	65.6
12	5.10	.104	2.64	.00849	5.48	4.72	4.81	8.30	8.47	29.9	52.7
13	3.99	.092	2.34	.00665	4.29	6.03	6.15	10.6	10.8	23.4	41.2
14	3.02	.080	2.03	.00503	3.24	7.97	8.13	14.0	14.3	17.7	31.2
15	2.44	.072	1.83	.00407	2.63	9.84	10.0	17.3	17.7	14.4	25.3
16	1.93	.064	1.63	.00322	2.07	12.5	12.7	21.9	22.4	11.3	20.0
17	1.48	.056	1.42	.00246	1.59	16.3	16.6	28.6	29.2	8.68	15.3
18	1.09	.048	1.22	.00181	1.17	22.1	22.6	39.0	39.7	6.38	11.2
19	0.754	.040	1.02	.00126	0.811	31.9	32.5	56.1	57.2	4.43	7.80
20	0.611	.036	0.914	.00102	0.657	39.4	40.2	69.3	70.7	3.59	6.31
21	0.482	.032	0.813	.000804	0.519	49.8	50.8	87.7	89.4	2.83	4.99
22	0.370	.028	0.711	.000616	0.397	65.1	66.4	114.5	117	2.17	3.82
23	0.271	.024	0.610	.000452	0.292	88.5	90.2	156	159	1.59	2.81
24	0.228	.022	0.559	.000380	0.245	105	107	186	189	1.34	2.36
25	0.189	.020	0.508	.000314	0.203	128	130	225	229	1.11	1.95

The above Resistances and Weights are subject to a variation of 2 per cent.

9. Physical, Mechanical, and Electrical Properties of Aluminium and Copper.

By kind permission of the British Aluminium Company, Limited, the following three Tables are taken from their Catalogue.

Property.	Aluminium.	Copper.
Chemical Symbol	Al	Cu
Atomic Weight	27.1	63.3
Atomic Volume	10.6	7.2
Position in Electro-Chemical Series	10	24
Melting Point, °Centigrade	655	1,100
Melting Point, °Fahrenheit	1,210	2,010
Specific Heat (water = 1)	0.212	0.094
Thermal Conductivity (Silver = 100)	31.3	73.6
Electric Conductivity (Silver = 100)	59.5	97.5
Co-efficient of Linear Expansion per °C	0.0000245	0.0000170
Co-efficient of Linear Expansion per °F	0.0000136	0.0000094
Specific Gravity, Cast	2.68	8.78
Specific Gravity, Rolled or Drawn	2.71	8.89
Tensile Strength in lbs. per sq. in.	26,000–33,000	55,700–62,500
Tensile Strength in kgs. per sq. mm.	18.3–23.2	39–43.9
Ratio of Tensile Strength to Weight (Equal Sectional Area)	1.63	1
Ratio of Tensile Strength to Weight (Equal Electric Conductance)	2.73	1
Elastic Limit as % of Tensile Strength	70%	75%
Modulus of Elasticity in lbs. per sq. in.	9,800,000	16,000,000
Modulus of Elasticity in kgs. per sq. mm.	6,890	11,200
Specific Resistance in Microhms per Cm. Cu. at 68° F. (20° C.) — Soft	2.826	1.724
Specific Resistance in Microhms per Cm. Cu. at 68° F. (20° C.) — Hard	2.933	1.760
Specific Resistance in Microhms per Cm. Cu. at 0° C. (32° F.) — Soft	2.801	1.587
Specific Resistance in Microhms per Cm. Cu. at 0° C. (32° F.) — Hard	2.700	1.620
Ohms per mil foot at 68° F.	17.00	10.4
Resistance of Conductor 1000 yards long by 1 sq. in. Cross Section, Ohms — Soft	0.04005	0.02443
Resistance of Conductor 1000 yards long by 1 sq. in. Cross Section, Ohms — Hard	0.04158	0.02495
Co-efficient of increase of Resistance with Temperature — Per °C.	0.0032–0.0040	0.0043
Co-efficient of increase of Resistance with Temperature — Per °F.	0.0018–0.0022	0.0024
Weight per 1000 yards by 1 sq. in. Section, lbs.	3,524	11,550
Ratio of Conductivities for Equal Area — Hard	0.60	1.0
Ratio of Conductivities for Equal Area — Soft	0.61	1.0
Ratio of Conductivities for Equal Resistance — Hard	0.66	1.0
Ratio of Conductivities for Equal Resistance — Soft	0.64	1.0
Ratio of Diameters for equal Resistance	1.28–1.29	1.0
Ratio of Weights for Equal Area	1.0	3.3
Ratio of Weights for Equal Resistance	1.0	2.0

Weights of Metals compared with Water.

Water weighs . .	62.3 lbs. per cu. ft.	0.036 lbs. per cu. in
Aluminium (cast) weighs	160 ,, ,,	0.092 ,, ,,
Iron weighs . . .	480 ,, ,,	0.263 ,, ,,
Brass ,, . . .	510 ,, ,,	0.295 ,, ,,
Copper ,, . . .	550 ,, ,,	0.320 ,, ,,
Silver ,, . . .	655 ,, ,,	0.380 ,, ,,
Lead ,, . . .	710 ,, ,,	0.411 ,, ,,
Gold ,, . . .	1200 ,, ,,	0.700 ,, ,,

Melting Points and Expansion of Metals.

Metal.	Melting Point ° Fahr.	Co-efficient of Expansion per ° F.
Lead . . .	620	0.0000156
Aluminium . .	1210	0.0000130
Brass . . .	1740	0.0000105
Silver . . .	1820	0.0000106
Copper . . .	1940	0.0000093
Gold . . .	2110	0.0000082
Iron (cast) . .	2200	0.0000062

10. Table of Specific Resistances at 0° C. of Pure Metals in Microhms per Centimetre Cube, with Temperature Coefficients (α) between 0° C. and 100° C.

	Microhms per Centimetre Cube.	α
Mercury (Pure)	94.070	0.00098
Lead (Pressed)	20.038	0.00411
Thallium (Pressed) . . .	17.633	0.00398
Tin (Pressed)	13.048	0.00440
Platinum	10.917	0.00367
Palladium	10.219	0.00354
Cadmium	10.023	0.00419
Iron	9.065	0.00625
Nickel (Electrolytic) . . .	6.935	0.00618
Zinc	5 751	0.00406
Magnesium	4.355	0.00381
Aluminium	2.665	0.00435
Gold	2.197	0.00377
Copper (Electrolytic) . . .	1.561	0.00428
Silver (Electrolytic) . . .	1.468	0.00400

11. VOLUME SPECIFIC RESISTANCES (R_0) IN C.G.S. UNITS OF ALLOYS AT 0° C. AND TEMPERATURE COEFFICIENTS (α AT 15° C.

	R_0	α
Platinum Silver, Pt 33% Ag 66% .	31582	0.000243
Platinum Iridium, Pt 80% Ir 20% .	30896	0.000822
Platinum Rhodium, Pt 90% Rh 10% .	21142	0.00143
Gold-Silver, Au 90%	6280	0.00124
Manganese Steel, Mn 12% . . .	67148	0.00127
Nickel Steel, Ni 4.35% . . .	29452	0.00201
German Silver, $Cu_8Zn_3Ni_2$ . . .	29982	0.000273
Platinoid	41731	0.00031
Manganin, Cu 84% Mn 12% Ni 4% .	46678	0.0000
Silverine	2064	0.00285
Aluminium Silver, Al 94% . .	4641	0.00238
Aluminium Copper, Al 94% . .	2904	0.00381
Copper Aluminium, Cu 97% . .	8847	0.000897
Copper Nickel Aluminium . . .	14912	0.000645
Titanium Aluminium	3887	0.00290

12. TABLE OF DIELECTRIC CONSTANTS OF SOLIDS.

Substance.	Dielectric Constant. Air = 1.	Authority.
Glass:		
Double extra dense flint. Density = 4.5	9.896	J. Hopkinson
Light flint. Density = 3.2 . .	6.72	Do.
Very light flint. Density = 2.87 .	6.61	Do.
Hard crown. Density = 2.485 .	6.96	Do.
Sulphur	2.24	Faraday
	2.88	Wullner
	3.84	Boltzmann
	4.00	Curie
	2.94	Blondlot
Ebonite	2.05	Rosetti
	3.15	Boltzmann
	2.21	Schiller
	2.86	Elsas
India Rubber:		
Pure brown	2.12	Schiller
Vulcanized grey	2.69	Do.
Gutta percha	2.462	Gordon

12. TABLE OF DIELECTRIC CONSTANTS OF SOLIDS (*continued*).

Substance.	Dielectric Constant. Air = 1.	Authority.
Paraffin wax (white)	1.977	Gibson and Barclay
	2.32	Boltzmann
	2.29	Hopkinson
	1.99	Gordon
Shellac	2.95	Wullner
	2.74	Gordon
	3.04	Winklemann
Mica	6.64	Klemencic
	8.00	Curie
	7.98	Bouty
	5.97	Elsas
Quartz:		
Along optic axis	4.55	Curie
Perpendicular to axis	4.49	Do.
Ice at −23	78.0	Bouty

13. TABLE OF DIELECTRIC CONSTANTS OF LIQUIDS.

Substance.	Dielectric Constant. Air = 1.	Authority.
Water at 17° C.	80.88	Heerwagen
,, 25° C.	75.7	Rosa
,, 25.3° C.	78.87	Franke
Olive oil	3.16	Hopkinson
Castor oil	4.78	,,
Turpentine	2.15	Silow
,,	2.23	Hopkinson
Petroleum	2.072	Silow
,,	2.07	Hopkinson
Ethylic alcohol at 25° C.	25.7	Rosa
Ethylic ether	4.37	Doule
,,	4.8	Bouty
Acetic acid	9.7	Franke
Formic acid at 15° C.	62.0	Fleming and Dewar
Glycerine ,,	56.0	,, ,, ,,
Methyl alcohol ,,	34.0	,, ,, ,,
Mononitrobenzol ,,	32.0	,, ,, ,,
Acetone ,,	21.85	,, ,, ,,
Ethyl nitrate ,,	17.72	,, ,, ,,
Amyl alcohol ,,	16.0	,, ,, ,,
Aniline ,,	7.51	,, ,, ,,
Carbon bisulphide at 15° C.	2.67	,, ,, ,,
Paraffin (light oil) ,,	2.00 (about)	

14. TABLE OF DIELECTRIC CONSTANTS OF GASES.
From Klemencic's results.
Vacuum = 1.

Gas.	Dielectric Constant.	Optical Refractive Index.
Air	1.000586	1.000293
Hydrogen . . .	1.000264	1.000139
Carbon dioxide . .	1.000984	1.000454
Carbonic oxide . .	1.000694	1.000335
Nitrous oxide . . .	1.001158	1.000516
Olefiant gas . . .	1.001558	1.000720
Marsh gas . . .	1.000952	1.000442
Carbon bisulphide . .	1.002900	1.001478
Sulphur dioxide . .	1.00954	1.000703
Ether	1.00744	1.00154

INDEX

Acceleration, unit of, 59
Algebraic Series, 16
Alternating current, penetration of, into a conductor, 78
Alternator, High Frequency, 227
Aluminium Wire, Resistances and Weights of, for Various Sizes, 336
American Morse Code, 258
Analysis, Vector, 42
Anderson, Bridge for Measurement of Inductance, 153
Anderson Bridge, theory of the, 154
Angular Measurement, 22
Antenna, capacity of an, 215
Antenna, predetermination of the capacity of, by calculation, 136
Antenna Radiation Resistance, Experimental Determination of, 206
Antennæ of various forms, radiation decrement of, 203
Antennæ, open and closed, radiation from, 200
Arc Transmitter, 224
Areas of curves, rules for, 32
Arithmetic progression, 16
Austin, W. L., empirical formula of, for decrease of receiving antenna current with distance, 234

Beats given by coupled oscillation circuits, 169
Bellini and Tosi, directive antenna of, 201
Bessel's Equation, 41
Binomial Theorem, 18
Bjerknes, formula of, for determination of decrement, 117
Bridge Measurements, 146
Bridge Methods of Measurement, General Conditions for Balance in, 161

Calculus, Differential, the, 25
Calculus, Integral, the, 30
Capacities, rules for addition of, 135
Capacity Bridge, Fleming and Dyke, 141
Capacity Current of Condenser, 112
Capacity, determination of, by method of mixtures, 134
Capacity, electrical, 124
Capacity, experimental determination of, 131
Capacity measured as a resistance, 133
Capacity of two concentric spherical surfaces, 125
Capacity of parallel plate condenser, 129
Capacity of two concentric cylinders, 128
Capacity of a horizontal wire, 128
Capacity of a vertical wire, 126
Capacity of a sphere, 125
Capacity, Predetermination of, 125
Capacity, unit of, 124
Carborundum detector, 244
Carey-Foster Bridge for Mutual Inductance Measurement, 156
Characteristic curve of crystal detector, 243, 244
Characteristic of logarithm, 21
Clark Cell, the, 151
Codes and signals in radio-telegraphy, 225
Coefficient of Coupling, 159

Commutator, Fleming and Clinton, for Capacity Measurement, 132
Complex Quantities, 10
Complex Quantities, addition of, 11
Complex Quantities, rules for division of, 13
Complex Quantities, rules for multiplication of, 12
Concentric Cylinders, capacity of, 128
Condenser, relation of cutrent and voltage in a circuit containing a, 116
Condensers, Energy Losses in, determination of, 183
Contact detectors, various forms of, 225
Copper Wire, Tables of Sizes, Resistances and Weights, 334
Coupling, coefficient of, 159
Crompton potentiometer, the, 151
Curl of a vector, definition of the, 45
Curves, Lengths of, formulæ for the, 32
Cymometer, the Fleming, 171
Cymometer Measurements, 163

Decrement, Measurement of, by Resonance Curves, 175
Decrement due to Radiation, 195
Decremeter, Marconi Company's, 181
Definitions of Electric Units, 71
Del operator, the, 45
de Sauty, Method of, for the Comparison of Capacities, 134
Detectors, 233
Detectors, comparison of, for sensitivity, 252
Detectors, various forms of, 240
Determinants, 3
Determinants, numerical, 4
Dielectric Conductivity, measurement of, 142
Dielectric Constants, Tables of, 339, 340, 341
Dielectrics, energy losses in, 138
Differential Calculus, the, 25
Differential coefficients, 27
Differential Equations, 36
Differentiation, rules for, 27
Differentials, Second and Higher, 34
Dimensions of Various Units and Quantities, 59, 60
Directive Antenna, Bellini and Tosi, 201
Directive Antenna, Marconi, 201
Divergence of a vector, 44
Divergence Theorem, the, 52
Dunwoody, H. H. C., 244
Dynamos and Alternators, care of, 267

Earth plate resistance, 215
Earth plate testing, 215, 216
Electric and Magnetic Units, 63
Electric and Magnetic Units, dimensions of, 64, 68
Electric waves, measurements of length of, 173
Electromagnetic Field, Maxwell's Theory of the, 185
Ellipse, equation to an, 2
Energy Balance Sheet for a Wireless Transmitter, 221
Energy Dissipation in Dielectrics, 137
Energy Losses in Condensers, 183
Energy Losses in Condensers, measurement of the, 141
Equation to a straight line, 1
Equations of the Electromagnetic Field, 186
Equations, Linear, 2
Equations, Differential, 36
Expansion of Functions in Series, 35
Exponential expressions for the Sine and Cosine, 35
Exponential Theorem, the, 19

Fessenden Heterodyne Receiver, 250
Field around a Hertzian Oscillator, 190
Fleming, J. A., 105, 171, 248
Fleming Cymometer, the (frontispiece), 171
Fleming Hot Wire Ammeter, the, 105
Fleming Oscillation Valve, the, 248
Formula for determining wavelengths, 174
Formula for number of oscillations in a train, 165
Formula for time period of oscillation of condenser circuit, 164

Formula for oscillation frequency of a circuit having capacity and inductance, 122
Frequency Measurement, 167
Frequency, natural, of a condenser circuit, 121
Function, Mathematical, 1
Functions, Trigonometrical, 23

Gauss' Theorem, 47
Geometric Mean Distance, 92
Geometric Progression, 17
Glossary of Terms used in Radiotelegraphy, 270
Goldschmidt Alternator, explanation of action of, 231
Goldschmidt High Frequency Alternator, 229
Golschmidt tone wheel receiver, 251
Gradient of a mathematical quantity, 43

Henry the unit of inductance, 87
Hertzian Oscillator, field around a, 190
Hertzian Oscillator, theory of the, 188
Heterodyne Receiver, the, 250
High Frequency Alternators, 227
High Frequency Resistance, 77
High Frequency Resistance, measurement of, 85
High Frequency Resistance of Wires, formulæ for the, 82
High Frequency Voltage Measurement, 106
Horizontal Wire, capacity of a, 128
Hot Wire Ammeter, 104
Howe, G. W. O., 127
Howe, formula of, for antenna capacity, 136
Hyperbolic Functions, 52
Hyperbolic Functions, Table of, 330

Impedance, definition of, 113
Imperial and Metric Systems of Units, 56
Inductance, definition of, 87
Inductance, measurement of, 114, 153
Inductance of Circuits, 86
Inductance of two parallel wires, 70
Inductance, predetermination of, for various forms of circuit, 88
Induction Coils, care of, 267
Integral Calculus, 30
Integration, rules for, 31
International Morse Code, 256
Irrotational Vectors, 51

Kirchhoff's Corollaries, 146

Laplace's Operator, 46
Line, equation to a straight, 1
Line, Surface, and Volume Integrals, 51
Linear Equations, 2
Linear Equations, solution of, by determinants, 3, 6
Logarithmic decrement, 165
Logarithms, 20
Logarithms, rules for calculations with, 21
Logarithms, Tables of, 308, 309-324
Love, A. E. H., Diffraction of Electric Waves round a sphere, 235

Maclaurin's Theorem, 34
Magnetic Detector, Marconi, 241
Mantissa of logarithms, 21
Marconi Company Decremeter, 181
Marconi Company Tuner, 239
Marconi Company Wavemeter, 72
Marconi-Fleming Valve Receiver, 250
Marconi Magnetic Detector, 241
Mathematical Function, 1
Mathematical Notes, 1
Matthiessen's Standard of Copper Conductivity, 74
Maxwell's method for solution of problems in networks of conductors, 147
Maxwell's Theory of the Electromagnetic Field, 185
Measurement of Inductance, 174
Metals, resistivity of, 73
Metric and Imperial Units, 56
Modulus of complex quantities, 11
Modulus of logarithms, 20
Morse Code, American, 258
Morse Code, International, 256
Mutual Inductance, measurements of, 156

Mutual Inductance of two parallel wires, 91

Napierian Logarithms, 21
Napierian Logarithms, Table of, 324
Natural Frequency of oscillation of an Antenna in Series with a Tuning coil, Determination of the, 209
Networks of Conductors, problems concerning, 147
Notes, Mathematical, 1

Oscillation Circuits, 120, 163
Oscillation Constant, definition of, 123
Oscillation Frequency, 214
Oscillation Valve, Fleming, 248
Oscillation Valve, mode of using as a wave detector, 249
Oscillator, Hertzian, theory of the, 188

Parallel plates, capacity of two, 129
Parallel wires, capacity of, 127
Potential, the, 49
Potentials, Scalar and Vector, 50
Potentiometer, the, 150
Potentiometer, measurements of voltages by the, 152
Power factor, definition of, 114
Poynting's Theorem, proof of, 190
Practical Electric Units, 68
Practical Information for Wireless Operators, 255
Practical measurement of energy losses in condensers, 143

Radian, definition of a, 22
Radiant vector, the, 191
Radiation Decrement, the, 195
Radiation Decrement and Resistance, connection of the, 202
Radiation efficiency of an antenna, 204
Radiation from an antenna, distribution of, 237
Radiation from Closed Aerials, 198
Radiation from Wireless Antennæ, 194
Radiation resistance, definition of, 193
Radio-telegraphic Terms, Glossary of, 270
Rayleigh, Lord, formula for inductance of spiral circuit, 100
Reactance, definition of, 113
Receiving Circuits, 233
Rectangular circuit, inductance of a, 95
Rectifying Contacts, various, 245
Rectifying Detectors, 242
Resistance High Frequency, 77
Resistance of an Earth Plate, 214
Resistivities of Metals and Alloys, Tables of, 338, 339
Resistivity of Metals, Table of, 73
Resonance Curve, a, 167
Resonance Curves, delineation of, 175
Resonance Curves, measurements of decrement by, 177
Resonance, Electrical, 163
Root-Mean-Square Value of an Alternating Current, 101
Root-Mean-Square Value of a damped oscillation, 102

Scalar Product of Two Vectors, 15
Scalar and Vector Quantities, 9
Series, Algebraic, 16
Solids, volumes of, 33
Spark Frequency, 214
Spark Transmitters, 214
Spark Voltage, 107
Spark Voltages, Table of, 109, 110
Sphere, capacity of a, 125
Spiral Flat, inductance of a, 98
Spiral, inductance of a, 96
Square circuit, inductance of a, 95
Square Roots, Table of, 282
Squares, Tables of, 282
Standard Cells, 151
Stefan's formula for the inductance of a spiral circuit, 98
Stokes' Theorem, 51
Storage Cells, care of, 258
Straight wire, inductance of, 93
Summation of Series, 17
Systems of measurement, 56

Table of Dielectric Constants,
Table of Integrals, 31
Table of Logarithms, 309
Table of Natural Sines, Cosines and Tangents, 325
Table of Resistivities of Metals and Alloys, 338, 339

Table of Spark Voltages, 109
Table of Squares, Square Roots, Cubes and Cube Roots, 282
Table of Wire Gauges, 333
Theory of two coupled circuits having capacity and inductance, 170
Time Period of an Oscillation, formula for the, 120, 163
Time Period of Oscillation of a leaky Condenser Circuit, 166
Tone Wheel Receiver, 251
Transformers, care of, 267
Transmitter, energy losses in a, 218
Transmitter, Spark, 214
Trigonometrical Functions, 23
Trigonometrical Tables, 325, 330
Tuners, 238

Unit Vectors, Rules for Scalar and Vector Products of, 15, 16
Unitary Ratio, 70
Units, Absolute, 55
Units, Derived, 55
Units, Electric and Magnetic, 63
Units, Fundamental, 55
Units of length, mass and time, 56
Units, Practical Electric, 68
Units of Velocity and Acceleration, 63

Vector Analysis, 42
Vector and Scalar quantities, 9
Vector diagrams for condenser circuits, 118
Vector impedance of condenser circuit, 139
Vector potential, expressions for electric and magnetic force in terms of, 187
Vector product of two vectors, 18
Vectors, symbolical representation of, 10
Velocity, Units of, 59
Vertical Wire, capacity of a, 125

Wave-lengths, measurement of, 173
Weights of Copper Wire, 334, 335
Weston Cadmium Cell, the, 151
Wheatstone's Bridge, calculation of the currents in circuits of a, 148
Wheatstone's Bridge, Theory of the, 147
Wire Gauges, Table of, 333
Wireless Apparatus, care of, 267

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